Journal of Global Optimization **29:** 191–224, 2004. 191

A New Successive Partition Algorithm for Concave Minimization Based on Cone Decomposition and Decomposition Cuts

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(Received 22 February 2002; accepted in revised form 10 August 2003)

Abstract. In this paper we propose a new partition algorithm for concave minimization. The basic structure of the algorithm resembles that of conical algorithms. However, we make extensive use of the cone decomposition concept and derive decomposition cuts instead of concavity cuts to perform the bounding operation. Decomposition cuts were introduced in the context of pure cutting plane algorithms for concave minimization and has been shown to be superior to concavity cuts in numerical experiments. Thus by using decomposition cuts instead of concavity cuts to perform the bounding operation, unpromising parts of the feasible region can be excluded from further explorations at an earlier stage. The proposed successive partition algorithm finds an ε-global optimal solution in a finite number of iterations.

Key words. concave minimization, cone decomposition, decomposition cut, successive partition.

1. Introduction

In this paper we are concerned with the minimization of a concave function $f(x)$ with $f: \mathbb{R}^n \mapsto \mathbb{R}$ over a nonempty polytope P with $P \subset \mathbb{R}^n$, where $\mathbb{R}$ denotes the set of real numbers. Concave minimization is one of the best studied problems in global optimization. Apart from numerous direct applications in operations research, mathematical economics, engineering design, etc., there are several other types of global optimization problems, like bilinear programming, reverse convex programming, 0–1 quadratic programming, that can be transformed into equivalent concave minimization problems. Furthermore, techniques for solving concave minimization problems play a central role in global optimization. For details the reader is referred to Benson (1995, 1996), Horst and Tuy (1996) and Tuy (1998) and references therein.

Most of the difficulties with concave minimization problems arise because such problems may have a very large, even an exponentially large number of local optimal solutions (see, e.g., Kalantari 1986), and no local criteria are known that allow us to determine whether a local optimal solution is also a global one or not. Pardalos and Schnitger (1988) showed that even a problem as simple as minimizing a concave quadratic function over a hypercube is $\mathcal{NP}$-hard.

The methods for solving concave minimization problems fall mainly into three categories: enumerative methods, which also include cutting plane methods; successive partition methods; and successive approximation methods (see, e.g., Horst and Tuy, 1996). The successive partition methods are probably the most popular. The first successive partition algorithms were proposed by Bali (1973), and Zwart (1974) which are small modifications of a cone-covering algorithm proposed by Tuy (1964). Algorithms following the approach of Bali and Zwart are known as conical algorithms, whereas those algorithms following the approach of Falk and Soland (1969) are known as rectangular algorithms. Later, Horst (1976) proposed a third important successive partition algorithm, the simplicial algorithm.

To identify subregions in conical algorithms that do not contain solutions with an objective value smaller than the incumbent solution and can therefore be excluded from further explorations, Tuy (1964) introduced *concavity cuts*, also known as convexity cuts, intersection cuts and Tuy cuts. Other authors then applied these cuts in pure cutting plane algorithms for concave minimization (e.g., Cabot, 1974, Konno, 1976a, b; Horst and Tuy, 1996).

Recently the concavity cut concept has been extended by using cone decomposition to derive cutting planes, called decomposition cuts, which usually eliminate a much larger portion of the feasible region than concavity cuts (see Porembski, 1999). First numerical experiments have been quite encouraging. Therefore, one might expect that replacing concavity cuts by decomposition cuts in conical algorithms would also result in a substantial acceleration of these algorithms. In this paper we pursue this notion and propose a new successive partition algorithm based on the typical conical-approach concepts using cone decomposition and decomposition cuts.

This paper is structured as follows: In the next section we give a brief description of the basic operations of the successive partition algorithm. In Section 3 we introduce cone decomposition and decomposition cuts. Methods for subdividing a polytope into subpolytopes in the context of cone decomposition are discussed in Section 4. A finite successive partition algorithm for concave minimization is given in Section 5. Results of numerical experiments are reported in Section 6. Some remarks in Section 7 conclude the paper.

2. Basic Operations

In the following we consider the concave minimization problem

$$\min\{f(x) \mid x \in P\}, \tag{1}$$

where $f\colon \mathbb{R}^n \mapsto \mathbb{R}$ is concave on $\mathbb{R}^n$ and $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ is a polyhedron. For the sake of simplicity we assume that P is bounded with $\dim(P)=n$, i.e., P is a full-dimensional polytope, and that $f(x)$ is finite on $\mathbb{R}^n$ and not constant on P. Furthermore, we assume that the level sets

$$L(\gamma)=\{x\in\mathbb{R}^n \mid f(x)\geqslant\gamma\} \tag{2}$$

are closed and bounded for all real numbers γ. Note that since $f(x)$ is concave, the level sets $L(\gamma)$ are convex.

A well-known and useful feature of concave minimization problems is that there exists a vertex of P which is a global optimum (see Mangasarian, 1969, Theorem 5.2.3). Hence the search for a global optimum can be restricted to the vertices of P. In this context the concept of *star optimum* plays the role of local optimality. A star optimum is a vertex of P, for which no adjacent vertex attains a smaller objective value. Starting at an arbitrary vertex of P, a star optimum can be found by checking whether an adjacent vertex has a smaller objective value. If this is the case, then we go to the adjacent vertex with the smallest value, and repeat the procedure. Clearly, the process terminates at a star optimum.

Since the number of vertices of the polytope P is finite, the number of star optima is finite. Let $\varepsilon>0$ be a prescribed tolerance and let $\hat{x}_0$ be a star optimum. Therefore, an algorithm that determines a star optimum $\hat{x}_{k+1}$ at iteration $k+1$, starting with $k=0$, such that $f(\hat{x}_{k+1})<f(\hat{x}_k)-\varepsilon$, or establishes that such a vertex of P does not exist, terminates after a finite number of iterations with an ε-global optimal solution, i.e. a solution $\hat{x}\in P$ with $f(x)\geqslant f(\hat{x})-\varepsilon$ for all $x\in P$.

The crux of solving problem (1) with such an algorithm is to determine a point $\breve{x}\in P$ such that $f(\breve{x})<\hat{f}-\varepsilon$, where $\hat{f}$ is the value of the best solution known so far, or to establish that such a point does not exist. When we have found such a point it is not hard to determine a vertex of P with an objective value not larger than $f(\breve{x})$, and, starting with this vertex, we can find a star optimum $\hat{x}$ with $f(\hat{x})<\hat{f}-\varepsilon$. Therefore, in each iteration of the above algorithm we encounter the following subproblem, called the *core problem*:

$$\text{CORE}\quad \begin{array}{l}\text{Find } \breve{x}\in P \text{ with } \breve{x}\notin L(\hat{f}-\varepsilon)\\ \text{or establish P} \subset L(\hat{f}-\varepsilon).\end{array}$$

In this paper we propose a successive partition algorithm based on cone decomposition and decomposition cuts for solving this problem. The basic structure of the algorithm is as follows.

The starting point is a star optimum x_0 with $f(x_0)\geqslant\hat{f}$, i.e. $x_0\in\operatorname{int}(L(\hat{f}-\varepsilon))$, where $\operatorname{int}(\cdot)$ denotes the interior of a set. Since x_0 is a vertex of P, there exists an $(n,n+1)$-submatrix (A_0,b_0) of (A,b) such that A_0 is of full rank and $A_0x_0=b_0$. By defining

$$\begin{aligned}C(x_0)&:=\{x\in\mathbb{R}^n \mid A_0x\leqslant b_0\}\\ &=x_0+\operatorname{cone}(u_1,\dots,u_n),\end{aligned} \tag{3}$$

where $u_1, \ldots, u_n \in \mathbb{R}^n$ are directions of the edges of $C(x_0)$, we get a P-containing cone vertexed at x_0. Therefore, $C(x_0)$ provides an approximation of P. We decompose the cone $C(x_0)$ into 2^t suitable cones of dimension $n-t$, where t with $1 \leqslant t \leqslant n$ denotes the respective level of decomposition, such that the convex hull of these cones contains P. We then check whether one of the edges of the cones contains a point of P not contained in $L(\hat{f}-\varepsilon)$. If this is the case, then we have solved the core problem. Otherwise we derive a cutting plane with respect to the cones, the decomposition cut $d^{\mathsf{T}}x \geqslant \delta$, that eliminates only points in $P \cap \operatorname{int}(L(\hat{f}-\varepsilon))$. Let

$$\omega := \max\{d^{\mathsf{T}}x \mid x \in P\}, \quad \text{and} \quad x_\omega := \operatorname{argmax}\{d^{\mathsf{T}}x \mid x \in P\}.$$

Thus we have to distinguish among three cases:

Case 1. $\omega > \delta$, and $f(x_\omega) < \hat{f} - \varepsilon$.
Case 2. $\omega \leqslant \delta$.
Case 3. $\omega > \delta$, and $f(x_\omega) \geqslant \hat{f} - \varepsilon$.

If we have Case 1, then the core problem is solved since we have found a point $x_\omega \in P$ with $x_\omega \notin L(\hat{f}-\varepsilon)$. Case 2 implies $P \subset L(\hat{f}-\varepsilon)$ and therefore the core problem is also solved. If we have Case 3, then no statement is possible and further examinations are necessary. To this end we partition P into two subpolytopes P_1 and P_2 such that $P = P_1 \cup P_2$ and $P_1 \cap P_2 = \operatorname{bd}(P_1) \cap \operatorname{bd}(P_2)$, where $\operatorname{bd}(\cdot)$ denotes the boundary of a set. This is done in such a way that from the 2^t cones approximating P we can easily derive 2^t cones of dimension $n-t$ for P_1 and P_2, respectively.

After we have tried to further decompose the 2^t cones approximating P_i ($i = 1,2$) to get a higher level of decomposition t_i, i.e., $t \leqslant t_i$, we repeat the examination described above for the subpolytopes P_1 and P_2, i.e. for each of the polytopes P_1 and P_2 we derive a decomposition cut, and if for one of the subpolytopes, say P_1, we again have Case 3, then we partition the subpolytope P_1 into two subpolytopes $P_{1,1}$ and $P_{1,2}$, and so on.

The partition process terminates when for at least one subpolytope of P we have Case 1 or for all subpolytopes of P Case 2. By applying special rules in the subdivision process we ensure that we get only finite sequences of nested polytopes. Hence the partition algorithm for solving the core problem is finite, and so is the corresponding algorithm for solving the concave minimization problem.

3. Cone Decomposition and Decomposition Cuts

In this section we give a brief description of cone decomposition and decomposition cuts, and introduce some new concepts needed for the subdivision process.

3.1. PSEUDOVERTICES AND CONES

In this section we briefly describe cone decomposition and decomposition cuts. For details the reader is referred to Porembski (1999). Cone decomposition and decomposition cuts are extensions of the well-known concavity cut concept. A concavity cut is derived as follows.

As above, let x_0 be a star optimum with $f(x_0) \geqslant \hat{f}$, i.e. $x_0 \in \text{int}(L(\hat{f}-\varepsilon))$, and let x_0 be nondegenerate. We consider the cone $C(x_0) = x_0 + \text{cone}(u_1, \ldots, u_n)$ (see (3)) and in the first step determine the intersection points $E_i(\tau_i)$ of the cone edges $E_i(\tau) = x_0 + \tau u_i$, $\tau \geqslant 0$, with $\text{bd}(L(\hat{f}-\varepsilon))$. In the second step we determine a hyperplane $c^{\mathsf{T}}(x-x_0)=1$ containing these intersection points, i.e., $c^{\mathsf{T}}(E_i(\tau_i)-x_0)=1$ for $i=1,2,\ldots,n$. Since $P \subset C(x_0)$ and $L(\hat{f}-\varepsilon)$ is convex, with x_0 the *concavity cut* $c^{\mathsf{T}}(x-x_0) \geqslant 1$ eliminates only points of P contained in $\text{int}(L(\hat{f}-\varepsilon))$, i.e. it is a *valid cut*. To derive a concavity cut when x_0 is degenerate some small modifications are necessary (see, e.g., Benson, 1999).

A problem with this cut is that the cone $C(x_0)$ is, in general, a poor approximation of the polytope P. Hence the derived concavity cut may eliminate a large portion of $C(x_0) \cap \text{int}(L(\hat{f}-\varepsilon))$, but only a small portion of $P \cap \text{int}(L(\hat{f}-\varepsilon))$. To partially overcome this problem we decompose the n-dimensional cone $C(x_0)$ into 2^t cones of dimension $(n-t)$, where t denotes the level of decomposition, the cones being vertexed in $\text{int}(L(\hat{f}-\varepsilon))$, such that their convex hull contains P. This convex hull provides an improved approximation of P, allowing us to derive deeper cutting planes, called *decomposition cuts*.

For this purpose we extend the notion of a vertex. A vertex x_0 of $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ is a 0-dimensional face of P. This is equivalent to the conditions that $Ax_0 \leqslant b$ holds and that there exists an $(n, n+1)$-submatrix (A_0, b_0) of (A, b) such that A_0 is of full rank and $A_0 x_0 = b_0$, i.e. $x_0 = A_0^{-1} b_0$. By dropping the first condition we can extend this notion to a more general one, as in the following definition.

DEFINITION 3.1. Let $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ be a polyhedron with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $\dim(P) = n$, and let $Ax \leqslant b$ include no constraints $a_i^{\mathsf{T}} x \leqslant \beta_i$, $a_j^{\mathsf{T}} x \leqslant \beta_j$ with $(a_i^{\mathsf{T}}, \beta_i) = \lambda(a_j^{\mathsf{T}}, \beta_j)$ for some $\lambda \in \mathbb{R}^+ \setminus \{0\}$, where $\mathbb{R}^+$ denotes the set of nonnegative real numbers.

1. Let $(\tilde{A}, \tilde{b})$ be an $(n, n+1)$-submatrix of (A, b) such that $\tilde{A}$ is of full rank, and let $\tilde{x}$ be the unique solution of $\tilde{A}x = \tilde{b}$. $\tilde{x}$ is called a pseudovertex of P, and the set of pseudovertices of P is denoted by $\text{vert}^{\text{ps}}(P_{(A,b)})$.
2. If for $\tilde{x} \in \text{vert}^{\text{ps}}(P_{(A,b)})$ there exists one and only one $(n, n+1)$-submatrix $(\tilde{A}, \tilde{b})$ of (A, b) such that $\tilde{A}$ is of full rank and $\tilde{A}\tilde{x} = \tilde{b}$, then $\tilde{x}$ is called a nondegenerate pseudovertex. Otherwise $\tilde{x}$ is a degenerate pseudovertex.
3. If for $\tilde{x}_1, \tilde{x}_2 \in \text{vert}^{\text{ps}}(P_{(A,b)})$ there exist $(n, n+1)$-submatrices $(\tilde{A}_1, \tilde{b}_1)$, $(\tilde{A}_2, \tilde{b}_2)$ of (A, b) such that $\tilde{A}_1$ and $\tilde{A}_2$ are of full rank, $\tilde{A}_1\tilde{x}_1 = \tilde{b}_1$, $\tilde{A}_2\tilde{x}_2 = \tilde{b}_2$, and $(\tilde{A}_1, \tilde{b}_1)$ and $(\tilde{A}_2, \tilde{b}_2)$ differ in exactly one row, then $\tilde{x}_1, \tilde{x}_2$ are neighbors.

Note that the definitions of degeneracy, nondegeneracy and neighborhood of pseudovertices are similar to those generally used for vertices. Similar as for vertices of P, we can derive P-containing cones with respect to pseudovertices.

DEFINITION 3.2. Let $\tilde{x} \in \mathrm{vert}^{\mathrm{ps}}(P_{(A,b)})$ be nondegenerate with $\tilde{A}\tilde{x} = \tilde{b}$, where $(\tilde{A}, \tilde{b})$ is an $(n, n+1)$-submatrix of (A, b) such that $\tilde{A}$ is of full rank. The cone $C(\tilde{x})$ derived with respect to the pseudovertex $\tilde{x}$ is defined by

$$\begin{aligned} C(\tilde{x}) &:= \{x \in \mathbb{R}^n \mid \tilde{A}x \leqslant \tilde{b}\} \\ &= \tilde{x} + \mathrm{cone}(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n), \end{aligned}$$

where $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n$ are directions of the edges of $C(\tilde{x})$.

For the pseudovertices $\tilde{x}_1, \tilde{x}_2 \in \mathrm{vert}^{\mathrm{ps}}(P_{(A,b)})$, let us consider the corresponding cones

$$C(\tilde{x}_i) = \tilde{x}_i + \mathrm{cone}(\tilde{u}_{1_i}, \tilde{u}_{2_i}, \ldots, \tilde{u}_{n_i}) \qquad i = 1, 2$$

and let us denote by

$$E_{i,k} = \{\tilde{x}_i + \tau\tilde{u}_{k_i} \mid \tau \in \mathbb{R}^+\} \quad \text{and} \quad E^-_{i,k}(\tau) = \{\tilde{x}_i - \tau\tilde{u}_{k_i} \mid \tau \in \mathbb{R}^+\} \tag{4}$$

the kth edge of the cone $C(\tilde{x}_i)$ and its negative extension, respectively.

The following holds. $\tilde{x}_1$ and $\tilde{x}_2$ are neighbors if and only if $\tilde{x}_1$ lies on an edge or its negative extension of $C(\tilde{x}_2)$ and $\tilde{x}_2$ lies on an edge or its negative extension of $C(\tilde{x}_1)$. There are three possible cases, which lead us to the following types of neighborhood:

N_1-neighborhood: $\tilde{x}_1 \in E_{2,k}$ and $\tilde{x}_2 \in E_{1,k}$;
N_2-neighborhood: $\tilde{x}_1 \in E_{2,k} \wedge \tilde{x}_2 \in E^-_{1,k}$ or $\tilde{x}_1 \in E^-_{2,k} \wedge \tilde{x}_2 \in E_{1,k}$;
N_3-neighborhood: $\tilde{x}_1 \in E^-_{2,k}$ and $\tilde{x}_2 \in E^-_{1,k}$.

Note that $\tilde{x}_1$ and $\tilde{x}_2$ lie on a line $\mathrm{aff}(E_{1,k}) = \mathrm{aff}(E_{2,k})$, where $\mathrm{aff}(\cdot)$ denotes the affine hull. Then the N_1-, N_2- and N_3-neighborhoods can be interpreted as $\tilde{x}_1$ and $\tilde{x}_2$ 'facing each other', $\tilde{x}_1$ 'looking at the back' of $\tilde{x}_2$ or $\tilde{x}_2$ 'looking at the back' of $\tilde{x}_1$, and $\tilde{x}_1$ and $\tilde{x}_2$ 'lying back to back', respectively.

Any cone derived with respect to a pseudovertex of P contains P. Our goal is to choose a set of pseudovertices in such a way that we can reduce the dimension of the corresponding cones without losing information about the shape of the polytope P. To this end we introduce the following concepts.

DEFINITION 3.3. A set $S \subseteq \mathrm{vert}^{\mathrm{ps}}(P_{(A,b)})$ of nondegenerate pseudovertices containing no N_2-neighbors is called an N-set of $\mathrm{vert}^{\mathrm{ps}}(P_{(A,b)})$. For $\tilde{x} \in S$ we denote by $C_S(\tilde{x})$ the face of $C(\tilde{x})$ that is spanned by the vectors $\tilde{u}_k$ such that the edge

$E_k = \{\tilde{x} + \lambda \tilde{u}_k \mid \lambda \in \mathbb{R}^+\}$ and its negative extension $E_k^- = \{\tilde{x} - \lambda \tilde{u}_k \mid \lambda \in \mathbb{R}^+\}$ contain no pseudovertex in $S \setminus \{\tilde{x}\}$.

Note that a degenerate pseudovertex can be made nondegenerate by dropping appropriate constraints from $Ax \leqslant b$.

If $\tilde{x}_i \in S$ has no neighbors in S, then we have $C_S(\tilde{x}_i) = C(\tilde{x}_i)$. However, if $\tilde{x}_i$ has t neighbors in S with $0 < t \leqslant n$, then we have $\dim(C_S(\tilde{x}_i)) = n - t$. The construction of the cone $C_S(\tilde{x}_i)$ is motivated by the following theorem.

THEOREM 3.1 (Porembski, 1999, Theorem 3.1). *Let $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ be a polyhedron with $\dim(P) = n \geqslant 2$, and let $S = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_\ell\} \neq \emptyset$ be an N-set of $\mathrm{vert}^{\mathrm{ps}}(P_{(A,b)})$. Then we have*

$$P \subseteq \mathrm{conv}\Big(C_S(\tilde{x}_1), C_S(\tilde{x}_2), \ldots, C_S(\tilde{x}_\ell)\Big).$$

The quality of an approximation of P by $C_S(\tilde{x}_1), C_S(\tilde{x}_2), \ldots, C_S(\tilde{x}_\ell)$ depends on the choice of the N-set S. To construct an appropiate N-set S, we apply *cone decomposition*, which is described in the following subsection.

3.2. CONE DECOMPOSITION

As we have seen in Theorem 3.1, we can use the cones $C_S(\tilde{x}_i)$, $\tilde{x}_i \in S$, to approximate the polytope P. Our goal is to utilize these cones to derive deep cutting planes. For this purpose the N-set S has to be chosen in an appropriate way. To this end we will derive S in a series of steps. Starting with the N-set $S_0 := \{\tilde{x}_1\}$, we gradually enlarge S_0 such that $S_0 \subset S_1 \subset \cdots \subset S_q \subset \mathrm{int}(L(\hat{f} - \varepsilon))$, where $S_0, S_1, \ldots, S_q$ are N-sets of $\mathrm{vert}^{\mathrm{ps}}(P_{(A,b)})$. S_{i+1} is derived from S_i such that for $\tilde{x}_i \in S_i \cap S_{i+1}$ the cone $C_{S_{i+1}}(\tilde{x}_i)$ has one edge less than the cone $C_{S_i}(\tilde{x}_i)$.

To construct such N-sets we extend the notion of neighborhood of pseudovertices to cone edges. This is based on the following observation. Let S be an N-set of $\mathrm{vert}^{\mathrm{ps}}(P_{(A,b)})$, and let $\tilde{x}_1, \tilde{x}_2 \in S$ be neighbors. Then the corresponding $(n, n+1)$-submatrices $(\tilde{A}_1, \tilde{b}_1)$ and $(\tilde{A}_2, \tilde{b}_2)$ of full rank of (A, b) differ in only one row, i.e. there exists an $(n-1, n+1)$-matrix $(\breve{A}, \breve{b})$ that is a submatrix of $(\tilde{A}_1, \tilde{b}_1)$ and $(\tilde{A}_2, \tilde{b}_2)$.

For an edge $\overline{E}_1$ of the cone $C(\tilde{x}_1) = \{x \in \mathbb{R}^n \mid \tilde{A}_1 x \leqslant \tilde{b}_1\}$, $n-1$ constraints of $\tilde{A}_1 x \leqslant \tilde{b}_1$ are binding. If for $\overline{E}_1$ all $n-1$ constraints of $\breve{A}x \leqslant \breve{b}$ are binding, then $\overline{E}_1$ or its negative extension contains $\tilde{x}_2$. Thus in this case $\overline{E}_1$ is not an edge of $C_S(\tilde{x}_1)$. Hence for every edge of $C_S(\tilde{x}_1)$ $n-2$ constraints of $\breve{A}x \leqslant \breve{b}$ are binding. The same holds for the cone $C_S(\tilde{x}_2)$. This leads to the following definition.

DEFINITION 3.4. Let $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ be a polyhedron with $\dim(P) = n$, and let S be an N-set of $\mathrm{vert}^{\mathrm{ps}}(P_{(A,b)})$.

1. Let $\tilde{x}_1, \tilde{x}_2 \in S$ be neighbors. An edge $\overline{E}_1$ of $C_S(\tilde{x}_1)$ and an edge $\overline{E}_2$ of $C_S(\tilde{x}_2)$ are called neighbors if there exists an $(n-1, n+1)$-submatrix $(\breve{A}, \breve{b})$ of full

rank of (A,b) such that for $\overline{E}_1$ and $\overline{E}_2$ the same $n-2$ constraints of $\breve{A}x \leqslant \breve{b}$ are binding.

2. Let $S=\{\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_\ell\}$, and let $R_S=\{\overline{E}_1,\overline{E}_2,\ldots,\overline{E}_\ell\}$ be a set of cone edges, where $\overline{E}_i$ is an edge of $C_S(\tilde{x}_i)$. The set of cone edges R_S is N-isomorph if for every pair $\tilde{x}_{i_1},\tilde{x}_{i_2}\in S$ of neighbors the corresponding edges $\overline{E}_{i_1},\overline{E}_{i_2}\in R_S$ are also neighbors.

With the following proposition which is an extension of Theorem 4.2 in Porembski (1999) we lay the foundation for cone decomposition.

PROPOSITION 3.1. *Let $P=\{x\in\mathbb{R}^n \mid Ax\leqslant b\}$ be a polyhedron with* $\dim(P)=n$, *let $S=\{\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_\ell\}$ be an N-set of* $\mathrm{vert}^{\mathrm{ps}}(P_{(A,b)})$, *and let the set of cone edges* $\mathrm{R}_S=\{\overline{E}_1,\overline{E}_2,\ldots,\overline{E}_\ell\}$ *be N-isomorph. Furthermore, let $a_{j*}^{\mathsf{T}}x\leqslant\beta_{j*}$ and $a_{h*}^{\mathsf{T}}x\leqslant\beta_{h*}$ be constraints of $Ax\leqslant b$ such that for $i,k=1,2,\ldots,\ell$ the following hold*:

(A) $a_{j*}^{\mathsf{T}}\tilde{x}_i=\beta_{j*}$ *and* $a_{h*}^{\mathsf{T}}\tilde{x}_i\neq\beta_{h*}$;
(B) $\overline{E}_i\subseteq\{x\in\mathbb{R}^n \mid a_{j*}^{\mathsf{T}}x\leqslant\beta_{j*}\}$ *and* $\overline{E}_i\nsubseteq\{x\in\mathbb{R}^n \mid a_{j*}^{\mathsf{T}}x=\beta_{j*}\}$;
(C) *The hyperplane $a_{h*}^{\mathsf{T}}x=\beta_{h*}$ intersects $\overline{E}_i\cup\overline{E}_i^-$ at a point $\tilde{x}_{\ell+i}$, where if it intersects $\overline{E}_i$, then $a_{h*}^{\mathsf{T}}\tilde{x}_i<\beta_{h*}$; otherwise $a_{h*}^{\mathsf{T}}\tilde{x}_i>\beta_{h*}$*;
(D) *For $\tilde{x}_{\ell+i}$ exactly n constraints of $Ax\leqslant b$ are binding*;
(E) $\tilde{x}_{\ell+i}\neq\tilde{x}_{\ell+k}$ *for* $i\neq k$.

Let $S':=\{\tilde{x}_{\ell+1},\tilde{x}_{\ell+2},\ldots,\tilde{x}_{2\ell}\}$ and let $\widehat{S}:=S\cup S'$. Then the following hold:

1. $\widehat{S}$ *is an N-set of* $\mathrm{vert}^{\mathrm{ps}}(P_{(A,b)})$.
2. *For $\tilde{x}_i\in S$ the pseudovertex $\tilde{x}_{\ell+i}$ is the only neighbor in S', and for $\tilde{x}_{\ell+i}\in S'$ the pseudovertex $\tilde{x}_i$ is the only neighbor in S.*
3. *$\tilde{x}_i,\tilde{x}_k\in S$ are neighbors if and only if $\tilde{x}_{\ell+i},\tilde{x}_{\ell+k}\in S'$ are neighbors.*
4. $\dim(C_{\widehat{S}}(\tilde{x}_i))=\dim(C_{\widehat{S}}(\tilde{x}_{\ell+i}))=\dim(C_S(\tilde{x}_i))-1$ *for all* $\tilde{x}_i\in S$, $\tilde{x}_{\ell+i}\in S'$.

For the purpose of simplification an N-isomorph set can be seen as a collection of cone edges pointing in a similar direction. Then Proposition 3.1 can be interpreted as follows. Let there exist a hyperplane $a_{j*}^{\mathsf{T}}x=\beta_{j*}$, called a *base hyperplane*, containing all pseudovertices in S (see condition (A)), and let us consider an N-isomorph set of cone edges pointing in the direction a_{j*} (see condition (B)). Then a hyperplane $a_{h*}^{\mathsf{T}}x=\beta_{h*}$, called a *mirror hyperplane*, intersecting these cone edges in pairwise distinct points (see condition (E)) defines a new set of pseudovertices such that each of the ‘old’ pseudovertices is a neighbor of exactly one ‘new’ pseudovertex and vice versa.

The set of new pseudovertices mirrors the neighborhood relations in the set of old pseudovertices in the sense that two pseudovertices in the old set are neighbors if and only if the corresponding pseudovertices in the new set are neighbors. However, the neighborhood relation might change from N_1-neighborhood to N_3-neighborhood and vice versa. It is not hard to verify that if the new pseudovertices are nondegenerate (condition (D)), the set of new pseudovertices is also an N-set. Condition (C) ensures that we can merge the sets of old and new pseudovertices without getting N_2-neighborhood relations.

Based on the concepts introduced above, we derive an appropriate N-set S by the following procedure, where *depth* is a prechosen maximal decomposition depth, $\overline{E}_i = \{\tilde{x}_i + \lambda \tilde{u}_{i,j_i} \mid \lambda \in \mathbb{R}^+\}$ an edge of $C_{S_t}(\tilde{x}_i)$, and $\overline{E}_i^-$ its negative extension.

Cone Decomposition Procedure (CDP)

Set $S_0 := \{\tilde{x}_1\}$ with $\tilde{x}_1 := x_0$;
Set $deco := true$ and $t := 0$;
While ($deco$ and $t < depth$) do
  if there exists an N-isomorph set of cone edges $R_{S_t} = \{\overline{E}_1, \overline{E}_2, \ldots, \overline{E}_{2^t}\}$ and a constraint $a_{h_t^*}^{\mathsf{T}} x \leqslant \beta_{h_t^*}$ of $Ax \leqslant b$ such that for $i, k = 1, 2, \ldots, 2^t$, the following conditions hold:

1. $a_{h_t^*}^{\mathsf{T}} x = \beta_{h_t^*}$ intersects $\overline{E}_i \cup \overline{E}_i^-$ at a point $\tilde{x}_{2^t+i} \in \operatorname{int}(L(\hat{f} - \varepsilon))$;
2. if $a_{h_t^*}^{\mathsf{T}} x = \beta_{h_t^*}$ intersects $\overline{E}_i$, then $a_{h_t^*}^{\mathsf{T}} \tilde{x}_i < \beta_{h_t^*}$; otherwise $a_{h_t^*}^{\mathsf{T}} \tilde{x}_i > \beta_{h_t^*}$;
3. for $\tilde{x}_{2^t+i}$ exactly n constraints of $Ax \leqslant b$ are binding;
4. $\tilde{x}_{2^t+i} \neq \tilde{x}_{2^t+k}$ for $i \neq k$;

  then set $S_{t+1} := S_t \cup \{\tilde{x}_{2^t+1}, \tilde{x}_{2^t+2}, \ldots, \tilde{x}_{2^{t+1}}\}$ and $t := t+1$;
  else set $deco := false$;
Set $S := S_t$.

How CDP works is illustrated by the following simple example which is also taken from Porembski (1999).

EXAMPLE 3.1. Given a polyhedron P and a nondegenerate vertex x_0 of P with $x_0 \in \operatorname{int}(L(\hat{f} - \varepsilon))$, $L(\hat{f} - \varepsilon)$ has been omitted in Figure 1a, but the intersection points of the boundary of $L(\hat{f} - \varepsilon)$ and the edges of the respective cones are indicated by dots. In CDP we start with an N-set $S_0 = \{\tilde{x}_1\}$, where $\tilde{x}_1 := x_0$, and a cone $C_{S_0}(\tilde{x}_1) = \tilde{x}_1 + \operatorname{cone}(\tilde{u}_{1,1}, \tilde{u}_{1,2}, \tilde{u}_{1,3})$ (see Figure 1a). There exist three N-isomorph sets $R_{S_0}^j = \{E_{1,j}\}$ ($j = 1, 2, 3$). All these sets fulfill the if-conditions of CDP. We choose $R_{S_0}^3$ and the constraint that describes the right facet of P. By CDP we get an N-set $S_1 = \{\tilde{x}_1, \tilde{x}_2\}$ and the cones $C_{S_1}(\tilde{x}_1) = \tilde{x}_1 + \operatorname{cone}(\tilde{u}_{1,1}, \tilde{u}_{1,2})$ and $C_{S_1}(\tilde{x}_2) = \tilde{x}_2 + \operatorname{cone}(\tilde{u}_{2,1}, \tilde{u}_{2,2})$ (see Figure 1b). We have $P \subseteq \operatorname{conv}(C_{S_1}(\tilde{x}_1), C_{S_1}(\tilde{x}_2))$. There exist two N-isomorph sets $R_{S_1}^j = \{E_{1,j}, E_{2,j}\}$ ($j = 1, 2$). By choosing $R_{S_1}^2$ and the constraint describing the front facet of P we get $S_2 = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\}$ and $C_{S_2}(\tilde{x}_i) = \tilde{x}_i + \operatorname{cone}(\tilde{u}_{i,1})$ with $i = 1, 2, 3, 4$ (see Figure 1c). We have $P \subseteq \operatorname{conv}(C_{S_2}(\tilde{x}_1), C_{S_2}(\tilde{x}_2), C_{S_2}(\tilde{x}_3), C_{S_2}(\tilde{x}_4))$. There exists only one N-isomorph set $R_{S_2} = \{E_{1,1}, E_{2,1}, E_{3,1}, E_{4,1}\}$. Since there exists no P-describing

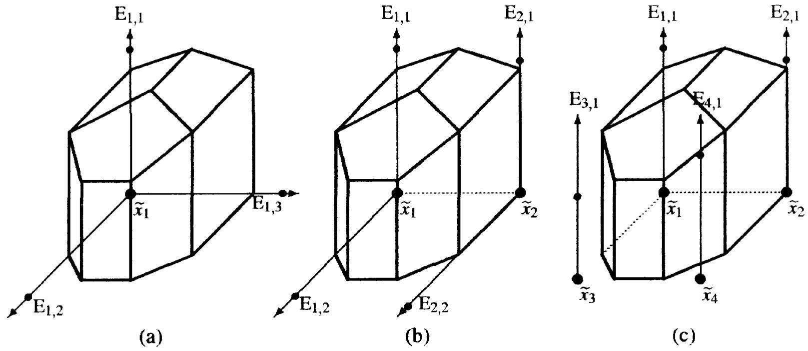

Figure 1. Decomposition of the cone $C(\tilde{x}_1)$ by CDP.

constraint that together with R_{S_2} fulfills the if-conditions of CDP, CDP stops with $S := S_2 \subseteq \mathrm{int}(L(\hat{f} - \varepsilon))$. □

In CDP we assume that the mirror-hyperplane-defining constraint $a_{h_t^*}^{\mathsf{T}} x \leqslant \beta_{h_t^*}$ is contained in the P-describing system $Ax \leqslant b$. But sometimes such a CDP-conditions-fulfilling constraint is not available. One way to define such a constraint is to determine a supporting hyperplane of P. Note that any supporting hyperplane $a^{\mathsf{T}} x = \beta$ of P is a positive linear combination of facets of P-describing hyperplanes, i.e. there exists $v \geqslant 0$ with $a = v^{\mathsf{T}} A$ and $\beta = v^{\mathsf{T}} b$. Therefore, to get a constraint $\hat{a}^{\mathsf{T}} x \leqslant \hat{\beta}$ fulfilling condition 1 to 4 in a first step we determine the intersection points $\overline{E}_i(\tau_{i,k}) := \tilde{x}_i + \tau_{i,k} \tilde{u}_{i,k}$ and $\overline{E}_i^-(\tau_{i,k}^-) := \tilde{x}_i - \tau_{i,k}^- \tilde{u}_{i,k}$ of $\overline{E}_i \in R_{S_t}$ and $\overline{E}_i^-$ with $\mathrm{bd}(L(\hat{f} - \varepsilon))$, respectively. In a second step we determine a solution $\hat{v} \geqslant 0$ of

$$\begin{matrix} v^{\mathsf{T}} A \overline{E}_i(\tau_i) \geqslant v^{\mathsf{T}} b \\ v^{\mathsf{T}} A \overline{E}_i^-(\tau_i^-) \leqslant v^{\mathsf{T}} b \end{matrix} \qquad \text{for } i = 1, 2, \ldots, 2^t,$$

if one exists, set $\hat{a} := \hat{v}^{\mathsf{T}} A$ and $\hat{\beta} := \hat{v}^{\mathsf{T}} b$, and check whether $\hat{a}^{\mathsf{T}} x \leqslant \hat{\beta}$ fulfills conditions 3 and 4 of CDP, which is, in general, the case.

In CDP the mirror constraint $a_{h_t^*}^{\mathsf{T}} x \leqslant \beta_{h_t^*}$ of Proposition 3.1 has to be explicitly chosen, whereas the base constraint $a_{j_t^*}^{\mathsf{T}} x \leqslant \beta_{j_t^*}$ is implicitly determined by the choice of the N-isomorph set R_{S_t}, which can be seen by the proof of the following proposition.

PROPOSITION 3.2. *Let $S_0 := \{\tilde{x}_1\} \subset \mathrm{vert}^{\mathrm{ps}}(P_{(A,b)})$ be the starting N-set of CDP and let $(\tilde{A}_1, \tilde{b}_1)$ be the corresponding $(n, n+1)$-submatrix of full rank of (A, b) such that $\tilde{A}_1 \tilde{x}_1 = \tilde{b}_1$. For the cones $C_{S_t}(\tilde{x}_i)$, $\tilde{x}_i \in S_t$, derived in the tth iteration of CDP we have*

$$C_{S_t}(\tilde{x}_i) = \{x \in \mathbb{R}^n \mid \tilde{A}_{1_t} x \leqslant \tilde{b}_{1_t}, G_{i_t} x = h_{i_t}\},$$

where $(\tilde{A}_{1_t}, \tilde{b}_{1_t})$ *is an* $(n-t, n+1)$*-submatrix of* $(\tilde{A}_1, \tilde{b}_1)$ *and* (G_{i_t}, h_{i_t}) *is an* $(t, n+1)$*-submatrix of* (A, b), *and* $(\tilde{A}_{1_t}^{\top}, G_{i_t}^{\top})$ *is of full rank.*

Proof. We prove the proposition by induction in t. For $t=0$ the proposition obviously holds. Suppose it also holds for a decomposition depth up to $t-1$, where $t \geqslant 1$. Then for $\tilde{x}_i \in S_{t-1} = \{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{2^{t-1}}\}$ we have $C_{S_{t-1}}(\tilde{x}_i) = \{x \in \mathbb{R}^n \mid \tilde{A}_{1_{t-1}} x \leqslant \tilde{b}_{1_{t-1}}, G_{i_{t-1}} x = h_{i_{t-1}}\}$. Suppose that $R_{S_{t-1}} := \{\overline{E}_1, \dots, \overline{E}_{2^{t-1}}\}$ is N-isomorph. For an edge $\overline{E}_i \in R_{S_{t-1}}$ exactly one constraint of $\tilde{A}_{1_{t-1}} x \leqslant \tilde{b}_{1_{t-1}}$ is not binding. Because of condition (B) in Proposition 3.1 we can assume without lost of generality (w.l.o.g.) that for $i = 1, 2, \dots, 2^{t-1}$ we have

$$\begin{aligned}\overline{E}_i = \{x \in \mathbb{R}^n \mid & \tilde{a}_{1_{t-1},1}^{\top} x \leqslant \tilde{\beta}_{1_{t-1},1}, \tilde{A}_{1_{t-1}\setminus\{1\}} x = \tilde{b}_{1_{t-1}\setminus\{1\}}, \\ & G_{i_{t-1}} x = h_{i_{t-1}}\},\end{aligned} \tag{5}$$

where $\tilde{a}_{1_{t-1},1}^{\top} x \leqslant \tilde{\beta}_{1_{t-1},1}$ denotes the first row of $\tilde{A}_{1_{t-1}} x \leqslant \tilde{b}_{1_{t-1}}$ and $\tilde{A}_{1_{t-1}\setminus\{1\}} x = \tilde{b}_{1_{t-1}\setminus\{1\}}$ denotes the system we obtain by eliminating this row from $\tilde{A}_{1_{t-1}} x = \tilde{b}_{1_{t-1}}$, i.e. $\tilde{a}_{1_{t-1},1}^{\top} x = \tilde{\beta}_{1_{t-1},1}$ plays the role of the base hyperplane. Furthermore, suppose that $a_{h_{t-1}^*}^{\top} x \leqslant \beta_{h_{t-1}^*}$ fulfills the CDP conditions, i.e. $a_{h_{t-1}^*}^{\top} x = \beta_{h_{t-1}^*}$ is the mirror hyperplane. The unique intersection point of the mirror hyperplane $a_{h_{t-1}^*}^{\top} x = \beta_{h_{t-1}^*}$ with $\overline{E}_i$ or $\overline{E}_i^-$ defines the pseudovertex $\tilde{x}_{2^{t-1}+i}$. Hence we have $a_{h_{t-1}^*}^{\top} \tilde{x}_{2^{t-1}+i} = \beta_{h_{t-1}^*}$, $\tilde{A}_{1_{t-1}\setminus\{1\}} \tilde{x}_{2^{t-1}+i} = \tilde{b}_{1_{t-1}\setminus\{1\}}$ and $G_{i_{t-1}}^{\top} \tilde{x}_{2^{t-1}+i} = h_{i_{t-1}}$.

By construction $(a_{h_{t-1}^*}, \tilde{A}_{l_{t-1}\setminus\{1\}}^{\top}, G_{i_{t-1}})^{\top}$ is an (n,n)-submatrix of A of full rank and $\tilde{x}_{2^{t-1}+i}$ is a nondegenerate pseudovertex. Nondegenerate pseudovertices $\tilde{x}_i$ and $\tilde{x}_j$ are neighbors if and only if $\tilde{x}_i$ lies on an edge or its negative extension of $C(\tilde{x}_j)$ and $\tilde{x}_j$ lies on an edge or its negative extension of $C(\tilde{x}_i)$. Hence for $i = 1, 2, \dots, 2^t$ we have

$$C_{S_t}(\tilde{x}_i) = \{x \in \mathbb{R}^n \mid \tilde{a}_{1_{t-1},1}^{\top} x = \tilde{\beta}_{1_{t-1},1}, \tilde{A}_{1_{t-1}\setminus\{1\}} x \leqslant \tilde{b}_{1_{t-1}\setminus\{1\}}, G_{i_{t-1}} x = h_{i_{t-1}}\} \tag{6}$$

and

$$C_{S_t}(\tilde{x}_{2^t+i}) = \{x \in \mathbb{R}^n \mid a_{h_{t-1}^*}^{\top} x = \beta_{h_{t-1}^*}, \tilde{A}_{1_{t-1}\setminus\{1\}} x \leqslant \tilde{b}_{1_{t-1}\setminus\{1\}}, G_{i_{t-1}} x = h_{i_{t-1}}\}, \tag{7}$$

which follows from Propositions 3.1.2 and 3.1.3. By defining $\tilde{A}_{1_t} := \tilde{A}_{1_{t-1}\setminus\{1\}}$, $\tilde{b}_{1_t} := \tilde{b}_{1_{t-1}\setminus\{1\}}$,

$$G_{i_t}^{\top} := (G_{i_{t-1}}^{\top}, \tilde{a}_{1_{t-1},1}) \qquad h_{i_t}^{\top} := (h_{i_{t-1}}^{\top}, \tilde{\beta}_{1_{t-1},1}) \tag{8}$$

for $i = 1, 2, \dots, 2^{t-1}$ and

$$G_{i_t}^{\top} := (G_{i_{t-1}}^{\top}, a_{h_{t-1}^*}) \qquad h_{i_t}^{\top} := (h_{i_{t-1}}^{\top}, \beta_{h_{t-1}^*}) \tag{9}$$

for $i = 2^{t-1}+1, 2^{t-1}+2, \dots, 2^t$ we have proved Proposition 3.2. □

By Proposition 3.2 we have $\tilde{A}_{1_t}\tilde{x}_i = \tilde{b}_{1_t}$, $G_{i_t}\tilde{x}_i = h_{i_t}$ for all $\tilde{x}_i \in S_t$. Since the system $\tilde{A}_{1_t}\tilde{x}_i = \tilde{b}_{1_t}$ is independent of the respective pseudovertex and the pseudovertices in S are nondegenerate, the following holds.

COROLLARY 3.1. *$\tilde{x}_i, \tilde{x}_j \in S_t$ are neighbors if and only if (G_{i_t}, h_{i_t}) and (G_{j_t}, h_{j_t}) differ in exactly one row.*

In the following proposition we describe some properties of the systems $G_{i_t}x = h_{i_t}$ that are useful for the subdivision methods discussed in the next section.

PROPOSITION 3.3. *Let $S_t := \{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{2^t}\}$ be an N-set of* $\mathrm{vert}^{\mathrm{ps}}(P_{(A,b)})$ *derived by CDP and let $C_{S_t}(\tilde{x}_i) = \{x \in \mathbb{R}^n \mid \tilde{A}_{1_t}x \leqslant \tilde{b}_{1_t}, G_{i_t}x = h_{i_t}\}$. For sets $S_{i_t,k}$ defined by $S_{i_t,k} := S_t \cap \{x \in \mathbb{R}^n \mid g_{i_t,k}^{\top}x = \theta_{i_t,k}\}$, where $g_{i_t,k}^{\top}x = \theta_{i_t,k}$ denotes the kth row of $G_{i_t}x = h_{i_t}$, the following hold.*

1. *$|S_{i_t,k}| = 2^{t-1}$ for $i = 1, 2, \dots, 2^t$ and $k = 1, 2, \dots, t$.*
2. *For $\tilde{x}_i \in S_t$ there exists for each row of $G_{i_t}x = h_{i_t}$ exactly one pseudovertex $\tilde{x}_j \in S_t$, such that $G_{i_t}x = h_{i_t}$ and $G_{j_t}x = h_{j_t}$ differ only in this row.*
3. *If $G_{i_t}x = h_{i_t}$ and $G_{j_t}x = h_{j_t}$ differ only in the kth row, then we have $S_t = S_{i_t,k} \cup S_{j_t,k}$ and $S_{i_t,k} \cap S_{j_t,k} = \emptyset$, i.e. the sets $S_{i_t,k}$ and $S_{j_t,k}$ can be interpreted as sets containing base pseudovertices and mirror pseudovertices, respectively.*

Proof. In each iteration of CDP but the last we perform a cone decomposition by choosing an N-isomorph set R_{S_r} and a mirror hyerplane $a_{h_r^*}^{\top}x = \beta_{h_r^*}$. On one hand, each of the sets R_{S_r} contains a unique edge of the starting cone $C_{S_0}(\tilde{x}_1) = \tilde{x}_1 + \mathrm{cone}(\tilde{u}_{1,1}, \tilde{u}_{1,2}, \dots, \tilde{u}_{1,n})$. On the other hand, by choosing an edge E_{1,i_r} of $C_{S_0}(\tilde{x}_1)$ that is still an edge of $C_{S_r}(\tilde{x}_1)$, we have uniquely determined the set R_{S_r}. Hence we can uniquely describe the sequence of decompositions in CDP by a sequence of edges $E_{1,i_1}, E_{1,i_2}, \dots, E_{1,i_t}$ of $C_{S_0}(\tilde{x}_1)$ and corresponding mirror hyperplanes $a_{h_1^*}^{\top}x = \beta_{h_1^*}, a_{h_2^*}^{\top}x = \beta_{h_2^*}, \dots, a_{h_t^*}^{\top}x = \beta_{h_t^*}$, i.e. the sequence $(E_{1,i_r}, a_{h_r^*}^{\top}x = \beta_{h_r^*})_{r=1}^{t}$ uniquely determines the resulting N-set S_t. It is not hard to verify that each permutation of this sequence results in the same N-set S_t. Hence we can assume w.l.o.g. that the kth row of $G_{i_t}x = h_{i_t}$ was derived at the tth iteration (see (8) and (9)), i.e. $k = t$, and that $R_{S_{t-1}} = \{E_{1,1}, E_{2,1}, \dots, E_{2^{t-1},1}\}$. With this we now prove the first assertion of Proposition 3.3.

(1) According to (8) and (9), $g_{i_t,t}^{\top}x = \theta_{i_t,t}$ is identical either with $\tilde{a}_{1_{t-1},1}^{\top}x = \tilde{\beta}_{1_{t-1},1}$ or with $a_{h_{t-1}^*}^{\top}x = \beta_{h_{t-1}}$. In the first case we have $g_{i_t,t}^{\top}\tilde{x}_j = \theta_{i_t,t}$ for all $\tilde{x}_j \in S_{t-1}$ and in the second we have $g_{i_t,t}^{\top}\tilde{x}_j = \theta_{i_t,t}$ for all $\tilde{x}_j \in S_t \backslash S_{t-1}$. Since $a_{h_{t-1}^*}^{\top}x = \beta_{h_{t-1}^*}$ has to fulfill the conditions in CDP, we have $\tilde{a}_{1_{t-1},1}^{\top}\tilde{x}_j \neq \tilde{\beta}_{1_{t-1},1}$ for all $\tilde{x}_j \in S_t \backslash S_{t-1}$ and $a_{h_{t-1}^*}^{\top}\tilde{x}_j \neq \beta_{h_{t-1}^*}$ for all $\tilde{x}_j \in S_{t-1}$. This implies that either $S_{i_t,t} = S_{t-1}$ or $S_{i_t,t} = S_t \backslash S_{t-1}$ holds. Because of $|S_{t-1}| = |S_t \backslash S_{t-1}| = 2^{t-1}$ this proves Proposition 3.3.1.

(2) To prove Proposition 3.3.2 we consider the last row of $G_{i_t}x=h_{i_t}$. It follows from the definition of $G_{i_t}x=h_{i_t}$ in (8) and (9) that $G_{i_t}x=h_{i_t}$ and $G_{j_t}x=h_{j_t}$ differ only in the last row, where $j=2^{t-1}+i$ in the case of $\tilde{x}_i\in S_{t-1}$ and $j=i-2^{t-1}$ otherwise. Let us denote by $G_{i_t\setminus\{t\}}x=h_{i_t\setminus\{t\}}$ the system that we obtain by eliminating the tth row of $G_{i_t}x=h_{i_t}$. The set $\{x\in\mathbb{R}^n \mid \tilde{A}_{1_t}x=\tilde{b}_{1_t}, G_{i_t\setminus\{t\}}x=h_{i_t\setminus\{t\}}\}$ describes a line in $\mathbb{R}^n$, and the unique intersection points of the hyperplanes $g^{\mathsf{T}}_{i_t,t}x=\theta_{i_t,t}$ and $g^{\mathsf{T}}_{j_t,t}x=\theta_{j_t,t}$ with this line define $\tilde{x}_i$ and $\tilde{x}_j$, respectively. We have $G_{i_t\setminus\{t\}}\tilde{x}_k\neq h_{i_t\setminus\{t\}}$ for all $\tilde{x}_k\in S_t\setminus\{\tilde{x}_i,\tilde{x}_j\}$, because otherwise $\{\tilde{x}_i,\tilde{x}_j,\tilde{x}_k\}$ would contain at least one pair of N_2-neighbors, which is a contradiction since S_t is an N-set. Hence $\tilde{x}_j$ is the only pseudovertex in S_t such that $G_{i_t}x=h_{i_t}$, and $G_{j_t}x=h_{j_t}$ differ only in the last row. This proves Proposition 3.3.2.

(3) To prove the last part of the proposition we assume that w.l.o.g. $i<j$ and that $G_{i_t}x=h_{i_t}$ and $G_{j_t}x=h_{j_t}$ differ only in the last row. Hence we have $j=2^{t-1}+i$. $g^{\mathsf{T}}_{i_t,t}x=\theta_{i_t,t}$ is identical with $\tilde{a}^{\mathsf{T}}_{1_{t-1},1}x=\tilde{\beta}_{1_{t-1},1}$, and $g^{\mathsf{T}}_{j_t,t}x=\theta_{j_t,t}$ is identical with $a^{\mathsf{T}}_{h_t^*}x=\beta_{h_t^*}$. Hence we have $S_{i_t,t}=S_{t-1}$ and $S_{j_t,t}=S_t\setminus S_{t-1}$, which proves Proposition 3.3.3. □

3.3. DECOMPOSITION CUTS

When CDP stops we have an N-set $S=\{\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_{2^t}\}$ of $\mathrm{vert}^{\mathrm{ps}}(P_{(A,b)})$ such that the corresponding cones $C_S(\tilde{x}_i)$ are $(n-t)$-dimensional and vertexed in $\mathrm{int}(L(\hat{f}-\varepsilon))$. In the case of $t=n$ we have $C_S(\tilde{x}_i)=\tilde{x}_i\in\mathrm{int}(L(\hat{f}-\varepsilon))$ and by Theorem 3.1 we have $P\subset\mathrm{conv}(S)\subset\mathrm{int}(L(\hat{f}-\varepsilon))$. Hence in this case we have solved the core problem since we have verified that there exists no $\breve{x}\in P$ with $\breve{x}\notin L(\hat{f}-\varepsilon)$.

In the case of $t<n$ we want to derive a cutting plane that eliminates as large a portion of P as possible. As was proved in Porembski (1999), Proposition 5.1, a cutting plane $d^{\mathsf{T}}x\geqslant\delta$ that intersects for all $\tilde{x}_i\in S$, similar to a concavity cut, the edges of $C_S(\tilde{x}_i)$ in $L(\hat{f}-\varepsilon)$ is a valid cut, i.e. it eliminates no point in $P\setminus\mathrm{int}(L(\hat{f}-\varepsilon))$. Since our aim is to derive a deep cutting plane, we try to derive a cut that eliminates as much as possible of each of the cones $C_S(\tilde{x}_i)$. To this end we determine the barycenter

$$\bar{x}=\frac{1}{2^t}\sum_{i=1}^{2^t}\tilde{x}_i \tag{10}$$

of the pseudovertices in S and the ‘average’ direction

$$\bar{v}:=\frac{1}{2^t(n-t)}\sum_{i=1}^{2^t}\sum_{k=1}^{n-t}\frac{\tilde{u}_{i,k}}{\|\tilde{u}_{i,k}\|} \tag{11}$$

of the cone edges of $C_S(\tilde{x}_1),C_S(\tilde{x}_2),\ldots,C_S(\tilde{x}_{2^t})$. Then we determine the *decomposition cut* $d^{\mathsf{T}}x\geqslant\delta$ such that the corresponding hyperplane $d^{\mathsf{T}}x=\delta$ intersects

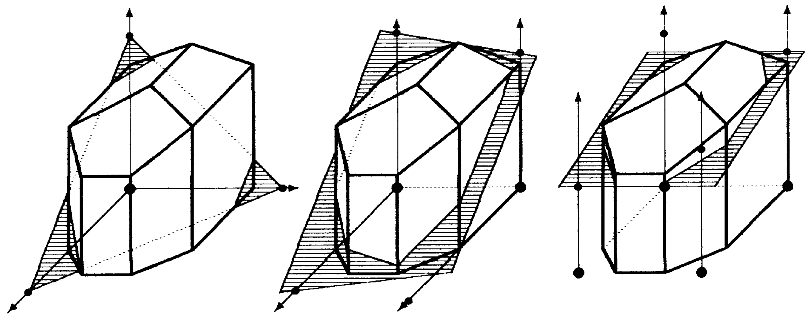

Figure 2. Decomposition cuts derived w.r.t. different decomposition depths.

all cone edges in $L(\hat{f}-\varepsilon)$ and thereby maximizes the distance between $\bar{x}$ and its intersection point with the ray $\bar{x}+\lambda\bar{v}$, $\lambda \geqslant 0$. This can be done with the following linear program.

$$\begin{array}{rll} \text{minimize} & d^{\mathsf{T}}\bar{v} & \\ \text{subject to} & -d^{\mathsf{T}}\bar{x}+\delta = 1 & \\ & d^{\mathsf{T}}\tilde{x}_i-\delta \leqslant -\rho & \text{for } i=1,2,\ldots,2^t \\ & d^{\mathsf{T}}E_{i,k}(\tau_{i,k})-\delta \geqslant 0 & \text{for } i=1,2,\ldots,2^t,\, k=1,2,\ldots,t, \end{array} \tag{12}$$

where $E_{i,k}(\tau_{i,k})$ denotes the intersection point of the kth edge of $C_{S_t}(\tilde{x}_i)$ with $\mathrm{bd}(L(\hat{f}-\varepsilon))$ and $\rho>0$ is a sufficiently small constant ensuring that the resulting cut eliminates all $\tilde{x}_i \in S$.

In general, the greater the decomposition depth t of cone decomposition, the deeper the corresponding decomposition cut turns out to be. This is also illustrated for the cases $t=1$, $t=2$, and $t=3$ in Figure 2 (see Example 3.1). Some results from computational experiments are reported in Porembski (1999).

4. Subdivision Methods

4.1. BASIC STRATEGIES

Let $S_0=\{\tilde{x}_1\}$ be the initial N-set of CDP, and let

$$C_{S_0}(\tilde{x}_1):=\{x\in\mathbb{R}^n \mid \tilde{A}_1 x\leqslant \tilde{b}_1\}. \tag{13}$$

Suppose that CDP decomposed the cone $C_{S_0}(\tilde{x}_1)$ into 2^t cones $C_{S_t}(\tilde{x}_i)$ with $\dim(C_{S_t}(\tilde{x}_i))=n-t$. According to Proposition 3.2, for $\tilde{x}_i\in S_t=\{\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_{2^t}\}$ we have

$$\begin{aligned} C_{S_t}(\tilde{x}_i) &= \tilde{x}_i+\mathrm{cone}(\tilde{u}_{i,1},\tilde{u}_{i,2},\ldots,\tilde{u}_{i,n-t}) \\ &= \{x\in\mathbb{R}^n \mid \tilde{A}_{1_t}x\leqslant \tilde{b}_{1_t}, G_{i_t}x=h_{i_t}\}, \end{aligned} \tag{14}$$

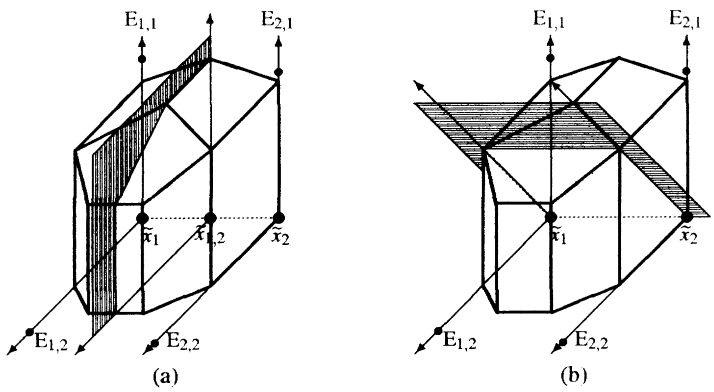

Figure 3. Subdivision between the cones and subdivision within the cones.

where $(\tilde{A}_{1_t}, \tilde{b}_{1_t})$ is an $(n-t, n+1)$-submatrix of $(\tilde{A}_1, \tilde{b}_1)$ and (G_{i_t}, h_{i_t}) is a $(t, n+1)$-submatrix of (A, b). Suppose that none of the edges of the cones $C_{S_t}(\tilde{x}_1), C_{S_t}(\tilde{x}_2), \ldots, C_{S_t}(\tilde{x}_{2^t})$ contains a point of $P \setminus L(\hat{f} - \varepsilon)$ and that for the decomposition cut derived with respect to these cones Case 3 of Section 2 holds, i.e. we have not found a solution with an objective value smaller than the best solution known so far, and we have not yet been able to verify $P \subset L(\hat{f} - \varepsilon)$.

For a further exploration we determine an appropriate hyperplane $p^\mathsf{T} x = \pi$ with $P \cap \{x \in \mathbb{R}^n \mid p^\mathsf{T} x < \pi\} \neq \emptyset$ and $P \cap \{x \in \mathbb{R}^n \mid p^\mathsf{T} x > \pi\} \neq \emptyset$ and subdivide P into subpolytopes P_1 and P_2 by defining

$$P_1 := P \cap \{x \in \mathbb{R}^n \mid p^\mathsf{T} x \leqslant \pi\} \quad \text{and} \quad P_2 := P \cap \{x \in \mathbb{R}^n \mid p^\mathsf{T} x \geqslant \pi\}. \tag{15}$$

To ensure that we can easily derive from the cones $C_{S_t}(\tilde{x}_i)$ used to approximate P cones that can be used for a good approximation of P_1 and P_2, respectively, we propose two methods for determining the partition hyperplane $p^\mathsf{T} x = \pi$.

In the first, called *subdivision between the cones*, we derive $p^\mathsf{T} x = \pi$ as a combination of two equalities contained in the systems $G_{i_t} x = h_{i_t}$, $i = 1, 2, \ldots, 2^t$ (see (14)), such that $p^\mathsf{T} x = \pi$ divides S_t into subsets $S_{i_{0_t},k}$ and $S_{j_{0_t},k}$ with $S_t = S_{i_{0_t},k} \cup S_{j_{0_t},k}$, $S_{i_{0_t},k} \cap S_{j_{0_t},k} = \emptyset$ and $p^\mathsf{T} x_i < \pi$ for all $x_i \in S_{i_{0_t},k}$ and $p^\mathsf{T} x_j > \pi$ for all $x_j \in S_{j_{0_t},k}$ (see Proposition 3.3). It follows from Proposition 3.3.2 that each $x_i \in S_{i_{0_t},k}$ has exactly one neighbor $x_j \in S_{j_{0_t},k}$. We derive new pseudovertices with corresponding cones by convex combinations of such neighboring pseudovertices and cones. This is illustrated in Figure 3a, where we have indicated the partition plane $p^\mathsf{T} x = \pi$, the new pseudovertex $\tilde{x}_{i,j}$, and the corresponding cone. Suppose that $S_t = \{\tilde{x}_1, \tilde{x}_2\}$. Then P_1 and P_2 can be approximated by the convex hull of $C_{S_t}(\tilde{x}_1)$ and the new cone and by the convex hull of $C_{S_t}(\tilde{x}_2)$ and the new cone, respectively.

In the second subdivision method proposed, called *subdivision within the cones*, we derive $p^\mathsf{T} x = \pi$ as a combination of two inequalities contained in the system $A_{1_t} x \leqslant b_{1_t}$ (see (14)). By doing so we partition P into subpolytopes P_1 and P_2 such that each cone $C_{S_t}(\tilde{x}_i)$, $\tilde{x}_i \in S_t$ is partitioned into two subcones. This is illustrated in Figure 3b. Then P_1 can be approximated by the convex hull of the collection of subcones contained in the halfspace $p^{\mathsf{T}}x \geqslant \pi$, and P_2 can be approximated by the convex hull of the collection of subcones contained in the halfspace $p^{\mathsf{T}}x \leqslant \pi$. In this method the N-set for the subpolytopes remains unchanged. In the following we discuss the two subdivision methods in detail.

4.2. SUBDIVISION BETWEEN THE CONES

In subdivision between the cones we make use of the properties of the systems $G_{i_t}x = h_{i_t}$, $i = 1, 2, \ldots, 2^t$, described in Proposition 3.3 to derive the partition hyperplane $p^{\mathsf{T}}x = \pi$. This is done as follows.

Step 1. Choose a pair of neighbors $\tilde{x}_{i_0}, \tilde{x}_{j_0} \in S_t$.
Step 2. Suppose that $G_{i_{0_t}}x = h_{i_{0_t}}$ and $G_{j_{0_t}}x = h_{j_{0_t}}$ differ in the kth row. Then define $S_{i_{0_t},k} := S_t \cap \{x \in \mathbb{R}^n \mid g^{\mathsf{T}}_{i_{0_t},k}x = \theta_{i_{0_t},k}\}$ and $S_{j_{0_t},k} := S_t \cap \{x \in \mathbb{R}^n \mid g^{\mathsf{T}}_{j_{0_t},k}x = \theta_{j_{0_t},k}\}$.
Step 3. Choose $\hat{\lambda}$ with $0 < \hat{\lambda} < 1$ and derive the partition hyperplane $p^{\mathsf{T}}x = \pi$ by defining $p := \hat{\lambda}(-g_{i_{0_t},k}) + (1-\hat{\lambda})g_{j_{0_t},k}$, and $\pi := \hat{\lambda}(-\theta_{i_{0_t},k}) + (1-\hat{\lambda})\theta_{j_{0_t},k}$.
Step 4. Define with $p^{\mathsf{T}}x = \pi$ the subpolytopes P_1 and P_2 according to (15).
Step 5. For all neighbors $\tilde{x}_i, \tilde{x}_j \in S_t$ with $\tilde{x}_i \in S_{i_{0_t},k}$ and $\tilde{x}_j \in S_{j_{0_t},k}$ determine

$$\lambda_{i,j} = \frac{\pi - p^{\mathsf{T}}\tilde{x}_j}{p^{\mathsf{T}}\tilde{x}_i - p^{\mathsf{T}}\tilde{x}_j},$$

set $\tilde{x}_{i,j} := \lambda_{i,j}\tilde{x}_i + (1-\lambda_{i,j})\tilde{x}_j$, and define $S_{(i_0,j_0)_t} := \{\tilde{x}_{i,j} \mid \tilde{x}_i \in S_{i_{0_t},k}, \tilde{x}_j \in S_{j_{0_t},k}$ are neighbors$\}$.
Step 6. Finally, set $S_t^{(1)} := S_{i_{0_t},k} \cup S_{(i_0,j_0)_t}$ and $S_t^{(2)} := S_{j_{0_t},k} \cup S_{(i_0,j_0)_t}$.

The following holds.

THEOREM 4.1. *Let the P_1- and P_2-describing systems $A_1x \leqslant b_1$ and $A_2x \leqslant b_2$ be obtained from $Ax \leqslant b$ by adding the constraints $p^{\mathsf{T}}x \leqslant \pi$ and $-p^{\mathsf{T}}x \leqslant -\pi$, respectively. Then $S_t^{(1)}$ is an N-set of* $\mathrm{vert}^{ps}(P_{1(A_1,b_1)})$ *and $S_t^{(2)}$ is an N-set of* $\mathrm{vert}^{ps}(P_{2(A_2,b_2)})$. *Furthermore, we have $S_t^{(1)}, S_t^{(2)} \subset \mathrm{int}(L(\hat{f} - \varepsilon))$ and*

$$\dim(C_{S_t^{(1)}}(\tilde{x}_i)) = \dim(C_{S_t^{(2)}}(\tilde{x}_j)) = n - t$$

for all $\tilde{x}_i \in S_t^{(1)}$, $\tilde{x}_j \in S_t^{(2)}$.

Proof. We prove Theorem 4.1 for $S_t^{(1)} \subset \mathrm{vert}^{ps}(P_{1(A_1,b_1)})$. For $S_t^{(2)}$ the assertions follow analogously. Since the P_1-describing system $A_1x \leqslant b_1$ was derived from the P-describing system $Ax \leqslant b$ by adding the constraint $p^{\mathsf{T}}x \leqslant \pi$, S_t is also an N-set of $\mathrm{vert}^{ps}(P_{1(A_1,b_1)})$. Therefore, $S_{i_{0_t},k}$ is also an N-set of $\mathrm{vert}^{ps}(P_{1(A_1,b_1)})$ since

$S_{i0_t,k} \subset S_t$ and any subset of an N-set is also an N-set. Furthermore, it follows from Proposition 3.3.2 that for each $\tilde{x}_i \in S_{i0_t,k}$ there is exactly one $\tilde{x}_j \in S_{j0_t,k}$ such that $\tilde{x}_i$ and $\tilde{x}_j$ are neighbors. Hence we have $\dim(C_{S_{i0_t,k}}(\tilde{x}_i)) = n-(t-1)$ for all $\tilde{x}_i \in S_{i0_t,k}$, and for each of the cones $C_{S_{i0_t,k}}(\tilde{x}_i)$ there is a uniquely determined cone edge $E_{i,k}$ such that the following holds. If $\tilde{x}_i \in S_{i0_t,k}$ and $\tilde{x}_j \in S_{j0_t,k}$ are N_1-neighbors, then $\tilde{x}_j$ lies on $E_{i,k}$, and if $\tilde{x}_i$ and $\tilde{x}_j$ are N_3-neighbors, then $\tilde{x}_j$ lies on $E^-_{i,k}$. The cone edges $E_{i,k}$ can be represented as follows.

$$E_{i,k} = \{x \in \mathbb{R}^n \mid \tilde{A}_{1_t} x = \tilde{b}_{1_t}, G_{i_t \setminus \{k\}} x = h_{i_t \setminus \{k\}}, g^{\mathsf{T}}_{i0_t,k}, x \leqslant \theta_{i0_t,k}\}, \tag{16}$$

where $g^{\mathsf{T}}_{i0_t,k} x \leqslant \theta_{i0_t,k}$ denotes the kth inequality in $G_{i0_t} x \leqslant h_{i0_t}$.

Let us assume w.l.o.g. $S_{i0_t,k} := \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{2^{t-1}}\}$. Then we can verify with (16) that the set of cone edges $R_{S_{i0_t,k}} := \{E_{1,k}, E_{2,k}, \ldots, E_{2^{t-1},k}\}$ is N-isomorph (see Definition 3.4 and Corollary 3.1). By construction the partition hyperplane $p^{\mathsf{T}} x = \pi$ intersects $E_{i,k} \in R_{S_{i0_t,k}}$ at $\tilde{x}_{i,j} \in \mathrm{conv}(\tilde{x}_i, \tilde{x}_j) \subset \mathrm{int}(L(\hat{f} - \varepsilon))$ if $\tilde{x}_i \in S_{i0_t,k}$ and $\tilde{x}_j \in S_{j0_t,k}$ are N_1-neighbors and it intersects $E^-_{i,k}$ at $\tilde{x}_{i,j} \in \mathrm{conv}(\tilde{x}_i, \tilde{x}_j) \subset \mathrm{int}(L(\hat{f} - \varepsilon))$ if $\tilde{x}_i$ and $\tilde{x}_j$ are N_3-neighbors, implying $p^{\mathsf{T}} \tilde{x}_i < \pi$ and $p^{\mathsf{T}} \tilde{x}_i > \pi$, respectively.

By setting $S := S_{i0_t,k}$, choosing $g^{\mathsf{T}}_{i0,k} x \leqslant \theta_{i0,k}$ as the base constraint $a^{\mathsf{T}}_{j*} x \leqslant \beta_{j*}$, $p^{\mathsf{T}} x \leqslant \pi$ as the mirror constraint $a^{\mathsf{T}}_{h*} x \leqslant \beta_{h*}$ and the N-isomorph set $R_S := \{E_{1,k}, E_{2,k}, \ldots, E_{2^{t-1},k}\}$ in Proposition 3.1 all assumptions of the proposition are fulfilled. By applying Proposition 3.1 we get the set $S^{(1)}_t$ defined in Step 6 with $S^{(1)}_t \subset \mathrm{int}(L(\hat{f} - \varepsilon))$, and the assertions of the proposition follow for $S^{(1)}_t$ from Propositions 3.1.1 and 3.1.4. □

By subdividing between the cones we partition P into subpolytopes P_1 and P_2 and for these subpolytopes derive N-sets $S^{(1)}_t$ and $S^{(2)}_t$, respectively. Let us consider the subpolytope P_1 with N-set $S^{(1)}_t$. We can derive a decomposition cut with respect to P_1 and $S^{(1)}_t$. If for this decomposition cut we face again Case 3 (see Section 2), then there are two options, partition P_1 again or to try to achieve a higher level of decomposition, i.e., to enlarge the N-set $S^{(1)}_t$. If we achieve a higher level of decomposition, if necessary by adding redundant constraints to the P_1-describing system, then we can derive a new decomposition cut that, in general, covers a larger portion of P_1 than the previous cut. However, if for this new decomposition cut we again face Case 3 and there is no way to achieve a higher level of decomposition, then we have to partition the subpolytope P_1 into smaller subpolytopes.

Hence by subdividing and decomposing we get sequences of nested subpolytopes

$$P := P^{(0)} \supset P^{(1)} \supset \cdots \supset P^{(r)}, \tag{17}$$

called *filters*, with corresponding N-sets $S^{(\ell)}_{t_\ell}$ of $\mathrm{vert}^{ps}(P^{(\ell)}_{(A^{(\ell)}, b^{(\ell)})})$, $\ell = 1, 2, \ldots, r$, where the subscript of $S^{(\ell)}_{t_\ell}$ indicates $|S^{(\ell)}_{t_\ell}| = 2^{t_\ell}$. To get finite convergence of such a successive partition process we have to ensure that only finite sequences of nested subpolytopes (17) are constructed.

To this end we choose $\tilde{x}_{i_0}^{(\ell)}, \tilde{x}_{j_0}^{(\ell)}$ in Step 2 and $\hat{\lambda}^{(\ell)}$ in Step 3 in such a way that after a finite number of subdivisions either one of the edges of the corresponding cones contains $\breve{x} \in P \setminus L(\hat{f} - \varepsilon)$, i.e. we have solved the core problem, or a further decomposition of the cones is possible, i.e. we can extend the N-set $S_{t_\ell}^{(\ell)}$ to an N-set $S_{t'_\ell}^{(\ell)}$ with $|S_{t'_\ell}^{(\ell)}| \geqslant 2|S_{t_\ell}^{(\ell)}|$. Therefore, if we do not identify an $\breve{x} \in P \setminus L(\hat{f} - \varepsilon)$, then after a finite number of subdivisions we get an N-set $S_{t_r}^{(r)}$ with $t_r = n$, i.e. $|S_{t_r}^{(r)}| = 2^n$ and $C_{S_{t_r}^{(r)}}(\tilde{x}_i) = \tilde{x}_i$ for all $\tilde{x}_i \in S_{t_r}^{(r)}$, which implies by Theorem 3.1 $P^{(r)} \subset \operatorname{conv}(S_{t_r}^{(r)}) \subset \operatorname{int}(L(\hat{f} - \varepsilon)$. Hence $P^{(r)}$ can be skipped from further explorations. Therefore, only finite filters are derived.

One way to choose $\tilde{x}_{i_0}^{(\ell)}, \tilde{x}_{j_0}^{(\ell)} \in S_{t_\ell}^{(\ell)}$ in Step 2 and $\hat{\lambda}^{(\ell)}$ in Step 3 is as follows: Choose $\tilde{x}_{i_0}^{(\ell)}$ and $\tilde{x}_{j_0}^{(\ell)}$ such that

$$\|\tilde{x}_{i_0}^{(\ell)} - \tilde{x}_{j_0}^{(\ell)}\| = \max\left\{\|\tilde{x}_i^{(\ell)} - \tilde{x}_j^{(\ell)}\| \mid \tilde{x}_i^{(\ell)}, \tilde{x}_j^{(\ell)} \in S_{t_\ell}^{(\ell)} \text{ are neighbors}\right\}, \tag{18}$$

where $\|\cdot\|$ denotes the Euclidian norm, and choose $\hat{\lambda}^{(\ell)}$ such that with $\rho_{j_0,k}^{(\ell)} := \theta_{j_{0_t},k} - g_{j_{0_t},k}^{\mathsf{T}} \tilde{x}_{i_0}^{(\ell)}$ and $\rho_{i_0,k}^{(\ell)} := \theta_{i_{0_t},k} - g_{i_{0_t},k}^{\mathsf{T}} \tilde{x}_{j_0}^{(\ell)}$

$$\frac{\delta \rho_{j_0,k}^{(\ell)}}{(1-\delta)\rho_{i_0,k}^{(\ell)} + \delta \rho_{j_0,k}^{(\ell)}} \leqslant \hat{\lambda}^{(\ell)} \leqslant \frac{(1-\delta)\rho_{j_0,k}^{(\ell)}}{\delta \rho_{i_0,k}^{(\ell)} + (1-\delta)\rho_{j_0,k}^{(\ell)}} \tag{19}$$

holds, where $\delta \in \mathbb{R}$ with $0 < \delta \leqslant \frac{1}{2}$ is a prechosen constant. Then we have $0 < \hat{\lambda}^{(\ell)} < 1$ and with increasing δ the lower bound in (19) increases and the upper bound decreases. For $\delta = \frac{1}{2}$ the lower and upper bounds are identical. (19) ensures that in Step 5

$$0 < \lambda_{i,j}^{(\ell)} < 1 \quad \text{and} \quad \delta \leqslant \lambda_{i_0,j_0}^{(\ell)} \leqslant 1 - \delta \tag{20}$$

hold which is necessary to ensure convergence of the algorithm as can be seen in the proof of the following lemma.

LEMMA 4.1. *Let* $P := P^{(0)} \supset P^{(1)} \supset \cdots \supset P^{(r)}$ *be a filter with N-sets* $S_{t_0}^{(0)}, S_{t_1}^{(1)}, \ldots, S_{t_r}^{(r)}$, *and let* $P^{(s)} \supset P^{(s+1)} \supset \cdots \supset P^{(r)}$ *be a subsequence such that the corresponding N-sets* $S_{t_s}^{(s)}, S_{t_{s+1}}^{(s+1)}, \ldots, S_{t_r}^{(r)}$ *contain the same number of pseudovertices, i.e.,* $t_s = t_{s+1} = \cdots = t_r$.

If in Step 2 $x_{i_0}^{(\ell)}, x_{j_0}^{(\ell)} \in S_t^{(\ell)}$ *and in Step 3* $\hat{\lambda}^{(\ell)}$, $\ell = s, s+1, \ldots, r$, *are chosen according to* (18) *and* (19), *respectively, then there exists* $\bar{x} \in L(\hat{f} - \varepsilon)$ *such that* $\lim_{r \to \infty} \|\tilde{x}_i^{(r)} - \bar{x}\| = 0$ *for all* $\tilde{x}_i^{(r)} \in S_{t_r}^{(r)}$.

Proof. Let us assume w.l.o.g. that we have $\tilde{x}_i^{(\ell+1)} = \tilde{x}_i^{(\ell)}$ and $\tilde{x}_j^{(\ell+1)} = \lambda_{i,j}^{(\ell)} \tilde{x}_i^{(\ell)} + (1 - \lambda_{i,j}^{(\ell)}) \tilde{x}_j^{(\ell)}$ (see Step 5). It follows from (20) that for all neighboring $\tilde{x}_i^{(\ell)}, \tilde{x}_j^{(\ell)} \in S_{t_\ell}^{(\ell)}$ with $\tilde{x}_i^{(\ell)} \in S_{i_{0_{t_\ell}},k_\ell}^{(\ell)}$ and $\tilde{x}_j^{(\ell)} \in S_{j_{0_{t_\ell}},k_\ell}^{(\ell)}$

$$\|\tilde{x}_i^{(\ell+1)} - \tilde{x}_j^{(\ell+1)}\| = (1 - \lambda_{i,j}^{(\ell)}) \|\tilde{x}_i^{(\ell)} - \tilde{x}_j^{(\ell)}\| < \|\tilde{x}_i^{(\ell)} - \tilde{x}_j^{(\ell)}\| \tag{21}$$

holds. Furthermore, we have

$$\|\tilde{x}_{i_0}^{(\ell+1)} - \tilde{x}_{j_0}^{(\ell+1)}\| \leqslant (1-\delta) \|\tilde{x}_{i_0}^{(\ell)} - \tilde{x}_{j_0}^{(\ell)}\|. \tag{22}$$

Since in $S_{t_\ell}^{(\ell)}$ we have exactly $\hat{c} := (n-t)2^{t-1}$ neighborhood relations, after at most $\hat{c}$ subdivisions we have

$$\begin{aligned} &\max\left\{ \|\tilde{x}_i^{(\ell+\hat{c})} - \tilde{x}_j^{(\ell+\hat{c})}\| \mid \tilde{x}_i^{(\ell+\hat{c})}, \tilde{x}_j^{(\ell+\hat{c})} \in S_{t_\ell}^{(\ell+\hat{c})} \text{ are neighbors} \right\} \\ &\quad \leqslant (1-\delta) \max\left\{ \|\tilde{x}_i^{(\ell)} - \tilde{x}_j^{(\ell)}\| \mid \tilde{x}_i^{(\ell)}, \tilde{x}_j^{(\ell)} \in S_{t_\ell}^{(\ell)} \text{ are neighbors} \right\}. \end{aligned}$$

This implies

$$\lim_{r\to\infty} \max\left\{ \|\tilde{x}_i^{(r)} - \tilde{x}_j^{(r)}\| \mid \tilde{x}_i^{(r)}, \tilde{x}_j^{(r)} \in S_{t_r}^{(r)} \text{ are neighbors} \right\} = 0. \tag{23}$$

Hence for neighboring $\tilde{x}_i^{(r)}, \tilde{x}_j^{(r)} \in S_{t_r}^{(r)}$ we have

$$\lim_{r\to\infty} \|\tilde{x}_i^{(r)} - \tilde{x}_j^{(r)}\| = 0. \tag{24}$$

Now let $\tilde{x}_i^{(\ell)}, \tilde{x}_j^{(\ell)} \in S_{t_\ell}^{(\ell)}$ be arbitrarily chosen. By construction of CDP there exists a 'path' $\tilde{x}_i^{(\ell)} =: \tilde{x}_{i_0}^{(\ell)}, \tilde{x}_{i_1}^{(\ell)}, \ldots, \tilde{x}_{i_{t_\ell}}^{(\ell)} := \tilde{x}_j^{(\ell)} \in S_{t_\ell}^{(\ell)}$ such that $\tilde{x}_{i_k}^{(\ell)}$ and $\tilde{x}_{i_{k+1}}^{(\ell)}$ are neighbors. In Figure 1c, for instance, $\tilde{x}_1, \tilde{x}_2, \tilde{x}_4$ defines such a path from $\tilde{x}_1$ to $\tilde{x}_4$. Because of (24) and $\|\tilde{x}_i^{(\ell)} - \tilde{x}_j^{(\ell)}\| \leqslant \sum_{k=1}^{t_\ell - 1} \|\tilde{x}_{i_k}^{(\ell)} - \tilde{x}_{i_{k+1}}^{(\ell)}\|$ we therefore have $\lim_{r\to\infty} \|\tilde{x}_i^{(r)} - \tilde{x}_j^{(r)}\| = 0$. Hence there exists $\bar{x}$ such that $\lim_{r\to\infty} \|\tilde{x}_i^{(r)} - \bar{x}\| = 0$.

Since $\bar{x} \in \operatorname{conv}(S_t^{(\ell)}) \subset \operatorname{int}(L(\hat{f} - \varepsilon))$ for $\ell = s, s+1, \ldots, r$, we have $\bar{x} \in L(\hat{f} - \varepsilon)$. □

We have just verified that in a subdivision process in which no further cone decomposition is performed the pseudovertices in $S_{t_\ell}^{(\ell)}$ converge to a point in $L(\hat{f} - \varepsilon)$. In a next step we want to extend these considerations to the extreme rays of the corresponding cones. To this end we introduce the following definition.

DEFINITION 4.1. A sequence of rays $E^{(\ell)} = \{x^{(\ell)} + \lambda u^{(\ell)} \mid \lambda \in \mathbb{R}^+\}$ is said to converge to the ray $\overline{E} = \{\bar{x} + \lambda \bar{u} \mid \lambda \in \mathbb{R}^+\}$ for $\ell \to \infty$, if

$$\lim_{\ell\to\infty} \|x^{(\ell)} - \bar{x}\| = 0 \quad \text{and} \quad \lim_{\ell\to\infty} \left\| \frac{u^{(\ell)}}{\|u^{(\ell)}\|} - \frac{\bar{u}}{\|\bar{u}\|} \right\| = 0.$$

The following holds:

LEMMA 4.2. *Under the assumptions of Lemma* 4.1 *let*

$$R^{(1)}_{S^{(\ell)}_{t_\ell}}, R^{(2)}_{S^{(\ell)}_{t_\ell}}, \ldots, R^{(n-t_\ell)}_{S^{(\ell)}_{t_\ell}} \quad \textit{with} \quad R^{(k)}_{S^{(\ell)}_{t_\ell}} = \{E^{(\ell)}_{1,k}, E^{(\ell)}_{2,k}, \ldots, E^{(\ell)}_{2^{t_\ell},k}\},$$

$k=1,2,\ldots,n-t_\ell$, *be the N-isomorph sets corresponding to* $P^{(\ell)}$ *and* $S^{(\ell)}_{t_\ell}$, $\ell=s,s+1,\ldots,r$. *If in Steps* 2 *and* 3 $x^{(\ell)}_{i_0}, x^{(\ell)}_{j_0} \in S^{(\ell)}_{t_\ell}$ *and* $\hat{\lambda}^{(\ell)}$ *are chosen according to* (18) *and* (19), *respectively, then there exist* $\bar{x}, \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{n-t_r} \in \mathbb{R}^n$ *with* $\bar{x} \in L(\hat{f}-\varepsilon)$ *such that the edges in* $R^{(k)}_{S^{(r)}_{t_r}}$ *converge to the ray* $\overline{E}_k = \{\bar{y}_k(\lambda) = \bar{x} + \lambda \bar{u}_k \,|\, \lambda \in \mathbb{R}^+\}$, $k=1,2,\ldots,n-t_r$ *for* $r \to \infty$.

Proof. Let us consider w.l.o.g. the N-isomorph set

$$R^{(1)}_{S^{(\ell)}_{t_\ell}} = \{E^{(\ell)}_{1,1}, E^{(\ell)}_{2,1}, \ldots, E^{(\ell)}_{2^{t_\ell},1}\},$$

and let us choose from this set two arbitrary neighboring edges, say $E^{(\ell)}_{i,1}$ and $E^{(\ell)}_{j,1}$. It follows from the definition of neighborhood of cone edges that there exists a two-dimensional affine subspace $\mathcal{A}_{i,j}$ of $\mathbb{R}^n$ containing these edges. Furthermore, let us define two lines $\mathcal{L}_1$ and $\mathcal{L}_2$, where the first contains $\tilde{x}^{(\ell)}_i$ and $\tilde{x}^{(\ell)}_j$ and the second is parallel to $\mathcal{L}_1$ and intersects the cone edges $E^{(\ell)}_{i,1}$ and $E^{(\ell)}_{j,1}$ at two points, say $E^{(\ell)}_{i,1}(1)$ and $E^{(\ell)}_{j,1}(1)$, with $E^{(\ell)}_{i,1}(1) \neq E^{(\ell)}_{j,1}(1)$.

Since after subdividing we define a new cone as a convex combination of the cones $C_{S_{t_\ell}}(\tilde{x}_i)$ and $C_{S_{t_\ell}}(\tilde{x}_j)$, the first edge of the new cone is a convex combination of $E^{(\ell)}_{i,1}$ and $E^{(\ell)}_{j,1}$ and is therefore also contained in $\mathcal{A}_{i,j}$. It is vertexed at $\tilde{x}_{i,j} := \lambda_{i,j}\tilde{x}_i + (1-\lambda_{i,j})\tilde{x}_j$ and intersects the line $\mathcal{L}_2$ at $\lambda_{i,j}E^{(\ell)}_{i,1}(1) + (1-\lambda_{i,j})E^{(\ell)}_{j,1}(1)$. With this construction it is not hard to verify that if $\lim_{r\to\infty} \|\tilde{x}^{(r)}_i - \tilde{x}^{(r)}_j\| = 0$, which follows from Lemma 4.1, then $\lim_{r\to\infty} \|E^{(r)}_{i,1}(1) - E^{(r)}_{j,1}(1)\| = 0$, implying that $E^{(r)}_{i,1}$ and $E^{(r)}_{j,1}$ converge to a ray. By using arguments similar to those in Lemma 4.1 we can see that we also have $\lim_{r\to\infty} \|E^{(r)}_{i,1}(1) - E^{(r)}_{j,1}(1)\| = 0$ for non-neighboring pseudovertices $\tilde{x}^{(r)}_i$ and $\tilde{x}^{(r)}_j$. Since $\lim_{r\to\infty} \|\tilde{x}^{(r)}_i - \tilde{x}^{(r)}_j\| = 0$ this implies that the edges in $R^{(1)}_{S^{(r)}_{t_r}}$ converge to a ray $\overline{E}_1\{\bar{x} + \lambda\bar{u}_1 \,|\, \lambda \geqslant 0\}$ for $r \to \infty$ □

With the last lemma we now have the tools to prove the following theorem.

THEOREM 4.2. *Under the assumptions of Lemmas* 4.1 *and* 4.2, *and under the assumption that we confine ourselves to finding an ε-global optimal solution, where* $\varepsilon > 0$, *let* $P := P^{(0)} \supset P^{(1)} \supset P^{(2)} \supset \cdots$ *be an arbitrary filter with corresponding N-sets* $S^{(0)}_{t_1}, S^{(1)}_{t_1}, S^{(2)}_{t_2}, \ldots$ *derived by the subdivision process described above. Then after a finite number of iterations* r_0 *one of the following cases holds*:

1. *We have identified a point* $\check{x} \in P^{(r_0)} \setminus L(\hat{f}-\varepsilon)$.

2. *We have verified* $P^{(r_0)} \subset L(\hat{f}-\varepsilon)$.
3. *A further decomposition of the cones derived w.r.t.* $P^{(r_0)}$ *and* $S^{(r_0)}_{t_{r_0}}$ *is possible.*

Proof. If for $P^{(r_0)}$ we face the first case in Theorem 4.2, then we have solved the core problem. There are two situations in which we might face the second case in Theorem 4.2. First, we have $|S^{(r_0)}_{t_{r_0}}| = 2^n$, which implies $P^{(r_0)} \subset \mathrm{conv}(S^{(r_0)}_{t_{r_0}}) \subset \mathrm{int}(L(\hat{f}-\varepsilon))$. Second, the decomposition cut derived w.r.t. $P^{(r_0)}$ and $S^{(r_0)}_{t_{r_0}}$ eliminates the complete subpolytope $P^{(r_0)}$, i.e. we face Case 2 of Section 1.

We prove Theorem 4.2 by contradiction. To this end let us assume that we have a filter $P := P^{(0)} \supset P^{(1)} \supset P^{(2)} \supset \cdots$ with corresponding N-sets $S^{(0)}_{t_0}, S^{(1)}_{t_1}, S^{(2)}_{t_2}, \ldots$ without encountering one of the three cases listed in Theorem 4.2. Hence, these sequences are infinite and there exists an index ℓ_0 such that $|S^{(\ell)}_{t_\ell}| = |S^{(\ell_0)}_{t_{\ell_0}}|$ for $\ell \geqslant \ell_0$. Therefore, for $\ell \geqslant \ell_0$ there exist $n - t_{\ell_0}$ N-isomorph sets $R^{(1)}_{S^{(\ell)}_{t_\ell}}, R^{(2)}_{S^{(\ell)}_{t_\ell}}, \ldots, R^{(n-t_{\ell_0})}_{S^{(\ell)}_{t_\ell}}$ with $R^{(k)}_{S^{(\ell)}_{t_\ell}} = \{E^{(\ell)}_{1,k}, E^{(\ell)}_{2,k}, \ldots, E^{(\ell)}_{2^{t_\ell},k}\}$, $k = 1, 2, \ldots, n - t_{\ell_0}$. According to Lemma 4.2 the rays in these sets converge to rays $\overline{E}_1, \overline{E}_2, \ldots, \overline{E}_{n-t_{\ell_0}}$, respectively.

Let us consider the ray $\overline{E}_1$. Since $\hat{f}$ is the objective value of the incumbent solution and we always check to see if a cone edge contains a point in P with a smaller objective value, none of the rays in $\mathrm{R}^{(1)}_{S^{(\ell)}_{t_\ell}} = \{E^{(\ell)}_{1,1}, E^{(\ell)}_{2,1}, \ldots, E^{(\ell)}_{2^{t_\ell},1}\}$, $\ell \geqslant \ell_0$, contains points of $P \setminus L(\hat{f})$. Hence this is also the case for $\overline{E}_1$. Let us denote by $\overline{E}_1(\bar{\lambda}_1)$ the intersection point of $\overline{E}_1$ with $L(\hat{f} - \frac{\varepsilon}{2})$. Then there exists a P-supporting hyperplane $\bar{a}^{\mathsf{T}}x = \bar{\beta}$ with $P \subset \{x \in \mathbb{R}^n \mid \bar{a}^{\mathsf{T}}x \leqslant \bar{\beta}\}$ that intersects $\overline{E}_1$ at $\overline{E}_1(\bar{\lambda}_1)$. Furthermore, since $f(x)$ is continuous and $\overline{E}_1(\bar{\lambda}_1) \in \mathrm{int}(L(\hat{f}-\varepsilon))$, there exists $\bar{\varrho}$ with $\bar{\varrho} > 0$ such that $B(\overline{E}_1(\bar{\lambda}_1), \bar{\varrho}) \cap \{x \in \mathbb{R}^n \mid \bar{a}^{\mathsf{T}}x = \bar{\beta}\}$ is contained in the interior of $L(\hat{f}-\varepsilon)$, where $B(y, \varrho)$ denotes the open ball with radius ϱ around y.

Since the rays in $R^{(1)}_{S^{(\ell)}_{t_\ell}}$ converge to the ray $\overline{E}_1$, there exists ℓ_1 with $\ell_1 \geqslant \ell_0$ such that each of the rays in $R^{(1)}_{S^{(\ell)}_{t_\ell}}$ intersects the hyperplane $\bar{a}^{\mathsf{T}}x = \bar{\beta}$ in $B(\overline{E}_1(\bar{\lambda}_1), \frac{\bar{\varrho}}{2})$, i.e. the intersection points are contained in $\mathrm{int}(L(\hat{f}-\varepsilon))$. Therefore, by using $\bar{a}^{\mathsf{T}}x = \bar{\beta}$ as a mirror hyperplane conditions 1. and 2. in CDP are fulfilled. In general, conditions 3, and 4. are also fulfilled, but if not we can ensure their fulfillment by shifting $\bar{a}^{\mathsf{T}}x = \bar{\beta}$ a little. Hence, a further cone decomposition is possible, which is a contradiction to the assumption. Hence we face after a finite number of partitions one of the cases stated in Theorem 4.2. □

To prove the finiteness of the subdivision process discussed in this subsection we had to confine ourselves to finding an ε-global optimal solution, where $\varepsilon > 0$. However, what happens when $\varepsilon = 0$?

THEOREM 4.3. *Let there be the same assumptions as in Theorem* 4.2. *If we are looking for an exact global optimum, i.e.* $\varepsilon = 0$, *then for an arbitrary filter*

$P := P^{(0)} \supset P^{(1)} \supset P^{(2)} \supset \cdots$ with corresponding N-sets $S^{(0)}_{t_0}, S^{(1)}_{t_1}, S^{(2)}_{t_2}, \ldots$ the following holds: Either the sequences are finite or the intersection point of a cone edge with the boundary of P converges to a point in $\mathrm{bd}(L(\hat{f}))$.

Proof. We prove Theorem 4.3 by contradiction. Suppose that we have an infinite sequence of subpolytopes $P := P^{(0)} \supset P^{(1)} \supset P^{(2)} \supset \cdots$ with corresponding N-sets $S^{(0)}_{t_0}, S^{(1)}_{t_1}, S^{(2)}_{t_2}, \ldots$. This implies that there exists an index ℓ_0 such that for $\ell \geqslant \ell_0$ none of the cases in Theorem 4.2 holds for any of the subpolytopes $P^{(\ell)}$ with corresponding N-sets $S^{(\ell)}_{t_\ell}$. Furthermore, let us assume that for none of the corresponding cones does the intersection point of a cone edge with the boundary of P converge to a point in $\mathrm{bd}(L(\hat{f}))$. Then there exists $\widetilde{\varepsilon}$ with $\widetilde{\varepsilon} > 0$ such that all intersection points of the cone edges with $\mathrm{bd}(P)$ are contained in $L(\hat{f} - \widetilde{\varepsilon})$. With arguments similar to those in the proof of Theorem 4.2 we can show that a further decomposition is possible, which is in contradiction to the assumption. □

According to Theorem 4.3, in the case of $\varepsilon = 0$ we get an infinite sequence of subpolytopes if and only if *all* N-isomorph sets converge to rays that intersect the boundary of P at a point lying on the boundary of $L(\hat{f})$. Such a situation might occur if we had only one N-isomorph set, but it is very unusual when there are more than one N-isomorph set, i.e. $t_\ell < n-1$. Since, in general, we do not have to decompose up to a level $n-1$ either to find a point $\check{x} \in P \setminus L(\hat{f})$ or to verify that a subpolytope is contained in $L(\hat{f})$, in the exact case, too, the subdivision process will usually terminate after a finite number of iterations. Finally, note that we get only infinite sequences of subpolytopes when the incumbent solution is actually a global optimum.

4.3. SUBDIVISION WITHIN THE CONES

For the second subdivision method proposed, subdivision within the cones, we assume that the cones $C_{S_t}(\tilde{x}_i)$, $\tilde{x}_i \in S_t$, are at least two-dimensional, i.e. $t \leqslant n-2$. To subdivide within the cones we define the partition hyperplane $p^{\mathsf{T}} x = \pi$ as a convex combination of two inequalities of the system $\tilde{A}_{1_t} x \leqslant \tilde{b}_{1_t}$, where

$$C_{S_t}(\tilde{x}_i) = \{x \in \mathbb{R}^n \mid \tilde{A}_{1_t} x \leqslant \tilde{b}_{1_t}, G_{i_t} x = h_{i_t}\}$$

and $\tilde{A}_{1_t} x \leqslant \tilde{b}_{1_t}$ is a subsystem of $Ax \leqslant b$ with $n-t$ inequalities (see Proposition 3.2). This is based on the following observations.

Let $\tilde{x}_i \in S_t$, and let $E_{i,1}$ and $E_{i,2}$ be arbitrarily chosen edges of the cone $C_t(\tilde{x}_i)$. Then there exist uniquely determined inequalities $\tilde{a}^{\mathsf{T}}_{1_t,1} x \leqslant \tilde{\beta}_{1_t,1}$ and $\tilde{a}^{\mathsf{T}}_{1_t,2} x \leqslant \tilde{\beta}_{1_t,2}$ in $\tilde{A}_{1_t} x \leqslant \tilde{b}_{1_t}$ such that for all $\tau > 0$ the following holds:

$$\begin{matrix} \tilde{a}^{\mathsf{T}}_{1_t,1} E_{i,1}(\tau) < \tilde{\beta}_{1_t,1}, \\ \tilde{A}_{1_t \setminus \{1\}} E_{i,1}(\tau) = \tilde{b}_{1_t \setminus \{1\}}, \end{matrix} \quad \text{and} \quad \begin{matrix} \tilde{a}^{\mathsf{T}}_{1_t,2} E_{i,2}(\tau) < \tilde{\beta}_{1_t,2}, \\ \tilde{A}_{1_t \setminus \{2\}} E_{i,2}(\tau) = \tilde{b}_{1_t \setminus \{2\}}. \end{matrix} \tag{25}$$

For the other edges $E_{i,j}$, $j=3,\dots,t$, it holds that

$$\tilde{a}_{1_t,1}^{\mathsf{T}} E_{i,j}(\tau) = \tilde{\beta}_{1_t,1} \quad \text{and} \quad \tilde{a}_{1_t,2}^{\mathsf{T}} E_{i,j}(\tau) = \tilde{\beta}_{1_t,2} \tag{26}$$

for all $\tau > 0$. When we define the partition hyperplane $p^{\mathsf{T}}x = \pi$ as

$$p := (-\tilde{\lambda})\tilde{a}_{1_t,1} + (1-\tilde{\lambda})\tilde{a}_{1_t,2} \tag{27}$$

and

$$\pi := (-\tilde{\lambda})\tilde{\beta}_{1_t,1} + (1-\tilde{\lambda})\tilde{\beta}_{1_t,2} \tag{28}$$

with $0 < \tilde{\lambda} < 1$ we therefore have

$$p^{\mathsf{T}} E_{i,1}(\tau) > \pi \quad \text{and} \quad p^{\mathsf{T}} E_{i,2}(\tau) < \pi \tag{29}$$

for $\tau > 0$, and

$$p^{\mathsf{T}} E_{i,j}(\tau) = \pi \tag{30}$$

for $\tau \geqslant 0$ and $j = 3,\dots,n-t$. Hence the partition hyperplane $p^{\mathsf{T}}x = \pi$ separates exactly two edges for each cone $C_{S_t}(\tilde{x}_i)$, $\tilde{x}_i \in S_t$, and contains the remaining edges. That is, we perform a simultaneous bisection of all cones. Based on these observations a subdivision within the cones can be performed as follows.

Step 1. Choose $\tilde{x}_i \in S_t$ and two edges $E_{i,1}$ and $E_{i,2}$ of the cone $C_{S_t}(\tilde{x}_i)$.

Step 2. Identify those inequalities $\tilde{a}_{1_t,1}^{\mathsf{T}} x \leqslant \tilde{\beta}_{1_t,1}$ and $\tilde{a}_{1_t,2}^{\mathsf{T}} x \leqslant \tilde{\beta}_{1_t,2}$ of $\tilde{A}_{1_t} x \leqslant \tilde{b}_{1_t}$ for which (25) holds.

Step 3. Choose $\tilde{\lambda}$ with $0 < \tilde{\lambda} < 1$ and derive the partition hyperplane $p^{\mathsf{T}}x = \pi$ according to (27) and (28).

Step 4. Define the subpolytopes $P_1 := P \cap \{x \in \mathbb{R}^n \mid p^{\mathsf{T}}x \leqslant \pi\}$ and $P_2 := P \cap \{x \in \mathbb{R}^n \mid p^{\mathsf{T}}x \geqslant \pi\}$ of P, and eliminate the inequalities $\tilde{a}_{1_t,2}^{\mathsf{T}} x \leqslant \tilde{\beta}_{1_t,2}$ and $\tilde{a}_{1_t,1}^{\mathsf{T}} x \leqslant \tilde{\beta}_{1_t,1}$ from the P_1- and P_2-describing systems, respectively, since these inequalities are now redundant.

Step 5. For all $\tilde{x}_i \in S_t$ define

$$\lambda_i := \frac{\pi - p^{\mathsf{T}} E_{i,2}(\bar{\tau}_{i,2})}{p^{\mathsf{T}} E_{i,1}(\bar{\tau}_{i,1}) - p^{\mathsf{T}} E_{i,2}(\bar{\tau}_{i,2})},$$

where $E_{i,j}(\bar{\tau}_{i,j})$ denotes the intersection point of $E_{i,j}$ with $\mathrm{bd}(L(\hat{f} - \varepsilon))$, and set $\bar{u}_i := (\lambda_i E_{i,1}(\bar{\tau}_{i,1}) + (1-\lambda_i) E_{i,2}(\bar{\tau}_{i,2})) - \tilde{x}_i$ for $i = 1,2,\dots,2^t$.

Step 6. For $\tilde{x}_i \in S_t$ the cones derived w.r.t. P_1 and P_2 are defined by $C_{S_t}(\tilde{x}_i) = \tilde{x}_i + \text{cone}(\tilde{u}_{i,1}, \bar{u}_i, \tilde{u}_{i,3}, \ldots, \tilde{u}_{i,n-t})$ and $C_{S_t}(\tilde{x}_i) = \tilde{x}_i + \text{cone}(\bar{u}_i, \tilde{u}_{i,2}, \ldots, \tilde{u}_{i,n-t})$, respectively, i.e. the directions $\tilde{u}_{i,1}$ and $\tilde{u}_{i,2}$ are replaced by the direction $\bar{u}_i$.

By construction the partition hyperplane $p^{\mathsf{T}}x = \pi$ contains all pseudovertices $\tilde{x}_i \in S_t$. To ensure that the pseudovertices in $S_{\underline{t}}$ remain nondegenerate, in Step 4 we have to eliminate the inequalities $\tilde{a}_{1_t,1}^{\mathsf{T}} x \leqslant \tilde{\beta}_{1_t,1}$ and $\tilde{a}_{1_t,2}^{\mathsf{T}} x \leqslant \tilde{\beta}_{1_t,2}$ from the P_1- and P_2-describing systems, respectively. Note that for P_1 and P_2 these inequalities are redundant.

In contrast to subdivision between the cones it is not possible to construct a finitely convergent partition algorithm (see Theorem 4.2) that uses only subdivisions within the cones. However, by choosing of $\tilde{x}_i \in S_t$, and $E_{i,1}$ and $E_{i,2}$ in Step 1 and $\tilde{\lambda}$ in Step 3 we can ensure that the resulting cones converge to rays. One way to do this is the following.

In the ℓth iteration choose $\tilde{x}_i^{(\ell)} \in S_{t_\ell}^{(\ell)}$ arbitrarily, and choose $E_{i,1}^{(\ell)}$ and $E_{i,2}^{(\ell)}$ such that

$$\begin{aligned} &|E_{i,1}^{(\ell)}(\bar{\tau}_{i,1}^{(\ell)}) - E_{i,2}^{(\ell)}(\bar{\tau}_{i,2}^{(\ell)})| \\ &\quad = \max\{|E_{i,k}^{(\ell)}(\bar{\tau}_{i,k}^{(\ell)}) - E_{i,h}^{(\ell)}(\bar{\tau}_{i,h}^{(\ell)})| \mid k, h = 1, 2, \ldots, n-t\}, \end{aligned} \tag{31}$$

i.e. the distance between the intersection points of $E_{i,1}^{(\ell)}$ with $\text{bd}(L(\hat{f} - \varepsilon))$ and $E_{i,2}^{(\ell)}$ with $\text{bd}(L(\hat{f} - \varepsilon))$ is maximal.

Let δ with $0 < \delta \leqslant \frac{1}{2}$ be a prescribed constant independent of the respective iteration and let $\tilde{\lambda}^{(\ell)}$ be chosen such that with $y_i^{(\ell)}(\delta) := \delta E_{i,1}^{(\ell)}(\bar{\tau}_{i,1}^{(\ell)}) + (1-\delta) E_{i,2}^{(\ell)}(\bar{\tau}_{i,2}^{(\ell)})$, $y_i^{(\ell)}(1-\delta) := (1-\delta) E_{i,1}^{(\ell)}(\bar{\tau}_{i,1}^{(\ell)}) + \delta E_{i,2}^{(\ell)}(\bar{\tau}_{i,2}^{(\ell)})$, $\hat{a} := \tilde{a}_{1_t,1} + \tilde{a}_{1_t,2}$ and $\hat{\beta} := \tilde{\beta}_{1_t,1} + \tilde{\beta}_{1_t,2}$

$$\frac{\tilde{a}_{1_t,2}^{\mathsf{T}} y_i^{(\ell)}(\delta) - \tilde{\beta}_{1_t,2}}{\hat{a}^{\mathsf{T}} y_i^{(\ell)}(\delta) - \hat{\beta}} \leqslant \tilde{\lambda}^{(\ell)} \leqslant \frac{\tilde{a}_{1_t,2}^{\mathsf{T}} y_i^{(\ell)}(1-\delta) - \tilde{\beta}_{1_t,2}}{\hat{a}^{\mathsf{T}} y_i^{(\ell)}(1-\delta) - \hat{\beta}} \tag{32}$$

holds, i.e. the resulting partition hyperplane intersects the line connecting $E_{i,1}^{(\ell)}(\bar{\tau}_{i,1}^{(\ell)})$ and $E_{i,2}^{(\ell)}(\bar{\tau}_{i,2}^{(\ell)})$ between $\delta E_{i,1}^{(\ell)}(\bar{\tau}_{i,1}^{(\ell)}) + (1-\delta) E_{i,2}^{(\ell)}(\bar{\tau}_{i,2}^{(\ell)})$ and $(1-\delta) E_{i,1}^{(\ell)}(\bar{\tau}_{i,1}^{(\ell)}) + \delta E_{i,2}^{(\ell)}(\bar{\tau}_{i,2}^{(\ell)})$. Note that for $\delta = \frac{1}{2}$ the lower and upper bound in (32) are identical and that in this case the subdivision of the cone $C_{S_{t_\ell}^{(\ell)}}(\tilde{x}_i)$ can be interpreted as an exact bisection (see Tuy, 1998). Also note that when two edges of a cone come 'closer together' by subdividing within the cones, then this is also the case for the corresponding edges of the other cones.

By using the usual concepts for verifying the exhaustiveness of a given conical subdivision process, described, for instance, in Horst and Tuy (1996), the following theorem can be proved.

THEOREM 4.4. *For any infinite sequence of subdivisions within the cones for which* $E_{i,1}^{(\ell)}$ *and* $E_{i,2}^{(\ell)}$ *are determined according to* (31) *and* $\tilde{\lambda}^{(\ell)}$ *is determined* *according to* (32), *for any cone* $C_{S^{(\ell)}_{i_\ell}}(\tilde{x}_i)$ *the distance between the intersection points of its cone edges with* $\mathrm{bd}(L(\hat{f}-\varepsilon))$ *converges to* 0, *i.e. the cones shrink to rays.*

Finally, note that since the edges of a cone belong to different N-isomorph sets, subdivision within the cones can be interpreted as a subdivision between two N-isomorph sets.

5. Partition Algorithm

In the proposed partition algorithm for the concave minimization problem (1) we do not derive a new partition of P for each core problem encountered but rather use a single-phase, successive partition algorithm. Note that when we have solved the kth core problem by identifying a solution with an objective value smaller than the best solution known so far, then for the constructed subpolytopes the following holds. All subpolytopes for which we face Case 2 of Section 2 in the kth core problem will initially also be allotted to Case 2 in the $(k+1)$th core problem. Furthermore, subpolytopes for which in the kth core problem Case 1 holds will initially be allotted to Case 3 in the $(k+1)$th core problem. Thus to solve the $(k+1)$th core problem we only have to examine the subpolytopes obtained by solving the kth core problem for which Case 1 or Case 3 holds. The basic structure of the proposed successive partition algorithm is as follows.

Successive Partition Algorithm (PARTI)

Compute a star optimum $x_0 \in P$;
Set $\hat{x} := x_0$ and $\hat{f} := f(x_0)$;
Set $\mathcal{P} := \{P\}$;
Set $S(P) := \{x_0\}$ (initial N-set of P);
While $\mathcal{P} \neq \emptyset$ do begin
  select $P' \in \mathcal{P}$;
  apply cone decomposition on P' and $S(P')$, if wanted and possible, resulting in an enlarged N-set $S(P')$;
  check whether one of the edges of the newly generated cones contains a point $x \in P'$ with $f(x) < \hat{f}$;
    If $x \in P'$ with $f(x) < \hat{f}$ was identified
    then begin
      find, starting with x, a star optimum x_0' with $f(x_0') \leqslant f(x)$;
      set $\hat{x} := x_0'$ and $\hat{f} := f(x_0')$;
    end;
  Derive a decomposition cut $d^{\mathsf{T}} x \geqslant \delta$ w.r.t. P' and $S(P')$;

Determine $\omega := \max\{d^{\mathsf{T}}x \mid x \in P'\}$, $x_\omega := \operatorname{argmax}\{d^{\mathsf{T}}x \mid x \in P'\}$;
If $\omega \geqslant \delta$ and $f(x_\omega) \leqslant \hat{f} - \varepsilon$
then begin
find, starting with x_ω, a star optimum x'_0 with $f(x'_0) < \hat{f}$;
set $\hat{x} := x'_0$ and $\hat{f} := f(x'_0)$;
end;
If $\omega < \delta$ then set $\mathcal{P} := \mathcal{P} \setminus \{P'\}$;
If $\omega \geqslant \delta$ and $f(x_\omega) \geqslant \hat{f} - \varepsilon$
then begin
subdivide P' into subpolytopes P'_1 and P'_2 by applying subdivision between the cones or subdivision within the cones;
determine the corresponding N-sets $S(P'_1)$ and $S(P'_2)$;
set $\mathcal{P} := \mathcal{P} \setminus \{P'\} \cup \{P'_1, P'_2\}$;
end;
end.

Four questions relating to PARTI must still be addressed.

1. How shall we select $P' \in \mathcal{P}$?
2. When and how long shall we apply cone decomposition?
3. When shall we apply subdivision between the cones and when subdivision within the cones?
4. How shall we choose $\hat{\lambda}$ in Step 3 when subdividing between the cones and $\tilde{\lambda}$ in Step 3 when subdividing within the cones to define the partition hyperplane?

In computational experiments we addressed these points as follows:

Ad 1: For each $P' \in \mathcal{P}$ with N-set $S(P')$ we determined the barycenter $x^*(P')$ of $S(P')$, and determined the point where the line connecting $x^*(P')$ and $x_\omega(P')$ intersects the corresponding decomposition cut. We selected that $P' \in \mathcal{P}$ for which the distance from this point to $x_\omega(P')$ is maximal.

Ad 2: We applied cone decomposition, which can become quite costly, only on those subpolytopes which were the fifth successors of a subpolytope on which cone decomposition has been applied.

Ad 3: We mainly applied subdivision between the cones. Subdivision within the cones was only applied when a subpolytope was overdue for a further decomposition.

Ad 4: We determined $\hat{\lambda}$ and $\tilde{\lambda}$ such that the resulting partition hyperplane contains $x_\omega(P') = \operatorname{argmax}\{d'^{\mathsf{T}}x \mid x \in P'\}$. However, if $\hat{\lambda}$ does not fulfill inequality

(19) or $\tilde{\lambda}$ does not fulfill inequality (32), where δ was chosen as $\delta = 0.2$, then set

$$\hat{\lambda} = \frac{\rho_{j_0,k}}{\rho_{i_0,k} + \rho_{j_0,k}} \text{ and } \tilde{\lambda} = \frac{\tilde{a}_{1_t,2}^{\mathsf{T}} y_i(\frac{1}{2}) - \tilde{\beta}_{1_t,2}}{\hat{a}^{\mathsf{T}} y_i(\frac{1}{2}) - \hat{\beta}}.$$

This corresponds to setting $\delta = \frac{1}{2}$.

EXAMPLE 5.1. To illustrate the working of PARTI let us consider the following three-dimensional concave minimization problem:

$$\min\{x^{\mathsf{T}} C x \mid Ax \leqslant b, x \geqslant 0\},$$

where

$$C = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}$$

(cf. Konno 1976b). The polytope $P = \{x \in \mathbb{R} \mid Ax \leqslant b, x \geqslant 0\}$ has eight vertices: $(0,0,0)$, $(2,0,0)$, $(0,2,0)$, $(0,0,2)$, $(1,1,1)$, $(\frac{12}{7}, \frac{6}{7}, 0)$, $(\frac{6}{7}, 0, \frac{12}{7})$, and $(0, \frac{14}{7}, \frac{6}{7})$. To illustrate in PARTI all subdivision strategies we allow a maximal decomposition depth of only 1. Furthermore, we set $\varepsilon = 0$.

In a first step we search for a star optimum. To this end let $(0,0,0)$ be the initial vertex in the search phase. Its objective value is 0. The neighbors are $(2,0,0)$, $(0,2,0)$ and $(0,0,2)$, all with an objective value of -8. We proceed to $(2,0,0)$. The neighbors of $(2,0,0)$ are $(0,0,0)$, $(\frac{12}{7}, \frac{6}{7}, 0)$, and $(\frac{6}{7}, 0, \frac{12}{7})$ with objective values of 0, $-\frac{216}{49}$, and $-\frac{360}{49}$, respectively. Hence, $\tilde{x}_1 = (2,0,0)$ is a star optimum and we have

$$\begin{aligned} C(\tilde{x}_1) &= \{x \in \mathbb{R}^3 \mid (3,1,2)x \leqslant 6,\ (0,0,-1)x \leqslant 0,\ (0,-1,0)x \leqslant 0\} \\ &= \tilde{x}_1 + \operatorname{cone}(\tilde{u}_{1,1}, \tilde{u}_{1,2}, \tilde{u}_{1,3}), \end{aligned}$$

where $\tilde{u}_{1,1} = (-1,0,0)$, $\tilde{u}_{1,2} = (-\frac{2}{7}, \frac{6}{7}, 0)$, and $\tilde{u}_{1,3} = (-\frac{8}{7}, 0, \frac{12}{7})$.

We set $\mathcal{P} = \{P\}$, $\hat{x} = (2,0,0)$ and $\hat{f} = -8$. We now enter the while-loop. We select $P' = P$, set $S(P') = \{\tilde{x}_1\}$ and apply cone decomposition. To decompose $C(\tilde{x}_1)$ we choose the N-isomorph set $R_{S(P')} = \{E_{1,3}\}$ according to the heuristic rules discussed in Porembski (1999), i.e. $(0,0,-1)x \leqslant 0$ is the base hyperplane, and we choose $(1,2,3)x \leqslant 6$ as mirror hyperplane. The mirror hyperplane intersects $E_{1,3} = \{\tilde{x}_1 + \lambda \tilde{u}_{1,3} \mid \lambda \geqslant 0\}$ at $\tilde{x}_2 = (\frac{6}{7}, 0, \frac{12}{7})$ which is a vertex of P with objective value $-\frac{360}{49}$. Hence we get $S(P') = \{\tilde{x}_1, \tilde{x}_2\}$,

$$\begin{aligned} C_{S(P')}(\tilde{x}_1) &= \{x \in \mathbf{R}^3 \mid (3,1,2)x \leqslant 6,\ (0,0,-1)x \leqslant 0,\ (0,-1,0)x = 0\} \\ &= \tilde{x}_1 + \operatorname{cone}(\tilde{u}_{1,1}, \tilde{u}_{1,2}) \end{aligned}$$

and

$$\begin{aligned}C_{S(P')}(\tilde{x}_2) &= \{x \in \mathbf{R}^3 \mid (3,1,2)x \leqslant 6,\ (0,0,-1)x \leqslant 0,\ (1,2,3)x = 6\}\\ &= \tilde{x}_2 + \operatorname{cone}(\tilde{u}_{2,1}, \tilde{u}_{2,2}),\end{aligned}$$

where $\tilde{u}_{2,1} = (-\frac{6}{7}, 0, \frac{2}{7})$ and $\tilde{u}_{2,2} = (\frac{1}{7}, 1, -\frac{5}{7})$.

We determine the intersection points of the edges of $C_{S(P')}(\tilde{x}_1)$ and $C_{S(P')}(\tilde{x}_2)$ with the boundary of $L(-8) = \{x \in \mathbb{R}^3 \mid x^{\mathsf{T}}Cx \geqslant -8\}$, and derive a decomposition cut $(-0.3889, 0.5444, 0.3889)x \geqslant 0.7778$ by solving the linear program (12). To check whether this cut eliminates the complete polytope P we solve the linear program $\max\{(-0.3889, 0.5444, 0.3889)x \mid x \in P\}$. The optimal solution is $x_\omega = (0, \frac{12}{7}, \frac{6}{7})$ with an objective value of 1.2667. Hence a subdivision of P' is due.

We start with a subdivision between the cones. The partition hyperplane $p^{\mathsf{T}}x = \pi$ is of the form

$$(\lambda \cdot (0,0,1) + (1-\lambda)\cdot(1,2,3))x = (1-\lambda)\cdot 6$$

(cf. Step 3 in Section 4.2). x_ω lies on $C_{S(P')}(\tilde{x}_2)$. Hence we must set

$$\lambda = \hat{\lambda} = \frac{6 - (1,2,3))\cdot \tilde{x}_1}{(0,0,1)\tilde{x}_2 + (6 - (1,2,3)\tilde{x}_1)} = \frac{4}{\frac{12}{7} + 4} = \frac{7}{10}$$

(cf. ad 4 in Section 5 and (19)). This gives us the partition hyperplane $(\frac{3}{10}, \frac{6}{10}, \frac{16}{10})x = \frac{18}{10}$, which is equivalent to $(3,6,16)x = 18$. With this we obtain the subpolytopes

$$\begin{aligned}&P_1 = P \cap \{x \in \mathbb{R}^3 \mid (3,6,16)x \leqslant 18\} \quad \text{and}\\ &P_2 = P \cap \{x \in \mathbb{R}^3 \mid (3,6,16)x \geqslant 18\}.\end{aligned}$$

The new pseudovertex defined by the partition hyperplane is given by $\tilde{x}_{1,2} = \frac{1}{2}\tilde{x}_1 + \frac{1}{2}\tilde{x}_2 = (\frac{10}{7}, 0, \frac{6}{7})$. With this we have for P_1 and P_2 the new N-sets

$$\begin{aligned}&S(P_1) = \{\tilde{x}_1^{[1]}, \tilde{x}_2^{[1]}\} \quad \text{with } \tilde{x}_1^{[1]} = \tilde{x}_1 \text{ and } \tilde{x}_2^{[1]} = \tilde{x}_{1,2} \text{ and}\\ &S(P_2) = \{\tilde{x}_1^{[2]}, \tilde{x}_2^{[2]}\} \quad \text{with } \tilde{x}_1^{[2]} = \tilde{x}_2 \text{ and } \tilde{x}_2^{[2]} = \tilde{x}_{1,2}.\end{aligned}$$

It holds that $C_{S(P_1)}(\tilde{x}_1^{[1]}) = C_{S(P')}(\tilde{x}_1)$, $C_{S(P_2)}(\tilde{x}_1^{[2]}) = C_{S(P')}(\tilde{x}_2)$ and

$$\begin{aligned}C_{S(P_i)}(\tilde{x}_2^{[i]}) &= \{x \in \mathbb{R}^3 \mid (3,1,2)x \leqslant 6,\ (0,-1,0)x \leqslant 0,\ (3,6,16)x = 18\}\\ &= \tilde{x}_2^{[i]} + \operatorname{cone}(\tilde{u}_{2,1}^{[i]}, \tilde{u}_{2,2}^{[i]})\end{aligned}$$

with $\tilde{u}_{2,1}^{[i]} = (-\frac{1}{16}, 0, 1)$, $\tilde{u}_{2,2}^{[i]} = (-\frac{4}{15}, \frac{42}{15}, -1)$ and $i = 1,2$. Finally we set $\mathcal{P} = \{P_1, P_2\}$.

In the next iteration we select P_1 and derive w.r.t. P_1 and $S(P_1)$ a decomposition cut $(-0.2825, 0.3954, 0.1147)x \geqslant 0.5649$. We have $\max\{(-0.2825, 0.3954, 0.1147)x \mid x \in P_1\} = 0.7909$ with optimal solution $(0,2,0)$. Hence P_1 must be explored further. We do this by subdividing P_1 within the cones. We must determine λ, such that the partition hyperplane $\lambda\cdot(3,1,2)-(1-\lambda)\cdot(0,0,-1))x = \lambda\cdot 6$ contains the vertex $(0,2,0)$ of P_1. This is the case for $\lambda = \frac{1}{3}$. Thus we obtain $(1,1,\frac{2}{3})x = 2$ as partition hyperplane. Hence we have

$$P_{1,1} = P_1 \cap \{x \in \mathbb{R}^3 \mid (1,1,\tfrac{2}{3})x \leqslant 2\},$$
$$P_{1,2} = P_1 \cap \{x \in \mathbb{R}^3 \mid (1,1,\tfrac{2}{3})x \geqslant 2\},$$

$\mathcal{P} = \{P_{1,1}, P_{1,2}, P_2\}$ and $S(P_{1,1}) = S(P_{1,2}) = S(P_1)$. We must replace the second directions by the directions $(-1,1,0)$ and $(-4,\frac{14}{3},-1)$, respectively, in the cones $C_{S(P_1)}(\tilde{x}_1^{[1]})$ and $C_{S(P_1)}(\tilde{x}_2^{[1]})$ to obtain the cones $C_{S(P_{1,1})}(\tilde{x}_j^{[1]})$, $j=1,2$, for approximating $P_{1,1}$. To obtain the cones for approximating $P_{1,2}$ the first directions must be replaced.

In the next iteration we select $P_{1,1} \in \mathcal{P}$ and derive w.r.t. $P_{1,1}$ and $S(P_{1,1})$ a decomposition cut $(-0.2825, 0.2825, 0.1147)x \geqslant 0.565$. $P_{1,1}$ is completely eliminated by this cut. Hence we have $\mathcal{P} = \{P_{1,2}, P_2\}$.

In the next iteration we select $P_{1,2} \in \mathcal{P}$ and derive the decomposition cut $(-0.0985, 0.3940, -0.1005)x \geqslant 0.7881$ which completely eliminates $P_{1,2}$, i.e. we have $\mathcal{P} = \{P_2\}$.

Therefore, in the next iteration only P_2 remains for selection. We derive the decomposition cut $(-0.3530, 0.9175, 0.8352)x \geqslant 1.6705$. We have $\max\{(-0.3530, 0.9175, 0.8352)x \mid x \in P_2\} = 0.7909$ with optimal solution $(0, \frac{12}{7}, \frac{6}{7})$. Hence P_2 is not completely eliminated by this cut and must be further subdivided. To this end we apply subdivision within the cones. The partition hyperplane obtained from the constraints $(3,1,2)x \leqslant 6$ and $(0,-1,0)x \leqslant 0$ must contain $(0, \frac{12}{7}, \frac{6}{7})$. We determine $\lambda = \frac{2}{5}$, which gives us the partition hyperplane $(\frac{6}{5}, 1, \frac{4}{5})x = \frac{12}{5}$ and is equivalent to $(6,5,4)x = 12$. We subdivide P_2 into

$$P_{2,1} = P_2 \cap \{x \in \mathbb{R}^3 \mid (6,5,4)x \leqslant 12\} \quad \text{and}$$
$$P_{2,2} = P_2 \cap \{x \in \mathbb{R}^3 \mid (6,5,4)x \geqslant 12\}.$$

The corresponding N-sets are $S(P_{2,1}) = S(P_{2,2}) = S(P_2)$. Thus we have $\mathcal{P} = \{P_{2,1}, P_{2,2}\}$.

In the next iteration we select $P_{2,1} \in \mathcal{P}$ and derive a decomposition cut $(-0.3530, 0.4965, 0.8352)x \geqslant 1.6700$. This cut completely eliminates $P_{2,1}$. Hence $\mathcal{P} = \{P_{2,2}\}$.

In the next iteration only $P_{2,2}$ remains for selection. We derive a decomposition cut $(-0.0532, 0.4399, 0.058)x \geqslant 0.9466$ which completely eliminates $P_{2,2}$. Hence we have $\mathcal{P} = \emptyset$, and the algorithm is terminated. $(0,2,0)$ with an objective value of -8 is a global optimum.

Note that we only allowed a maximal decomposition depth of 1. If we would have allowed a maximal decomposition depth of 2 or 3 the concave minimization problem would have been solved in the first iteration without the need for subdivisions. □

As we can see with Theorem 4.2, a successive partition algorithm using only subdivisions between the cones is finitely convergent as long as we decompose the cones whenever possible. However, the finite convergence of the algorithm can also be ensured by subdividing from time to time within the cones. In this way we can avoid cone decompositions that might be time-consuming when the actual level of decomposition is already deep without endangering the finite convergence of the partition algorithm. The following holds.

THEOREM 5.1. *In* PARTI *let the partition hyperplanes defining values of* $\hat{\lambda}^{(\ell)}$ *and* $\tilde{\lambda}^{(\ell)}$ *fulfill* (19) *and* (32). *Furthermore, after a finite number of subdivision within the cones let* PARTI *always perform a subdivision between the cones and vice versa. Then* PARTI *terminates after a finite number of iterations with an* ε*-global optimal solution.*

Proof. We prove Theorem 5.1 by contradiction. To this end let us assume that PARTI is infinite, i.e. there exists an infinite filter. Hence we have $P := P^{(0)} \supset P^{(1)} \supset \cdots$, and in this filter we have infinite sequences of subdivisions between the cones and within the cones. Furthermore, there exists ℓ_0 such that for all iterations $\ell \geqslant \ell_0$ no further cone decomposition is performed, i.e. the number of pseudovertices in $S_{t_\ell}^{(\ell)}$ remain constant for $\ell \geqslant \ell_0$, and a k_0 such that after iteration k_0 none of the edges of the newly generated cones contains a point in $P \setminus L(\hat{f})$. Note that the number of star optima is finite. Let w.l.o.g. $\ell_0 \geqslant k_0$.

For $\ell \geqslant \ell_0$ we always have $n - t_{\ell_0}$ N-isomorph sets. According to Lemma 4.1 and 4.2 the edges in the $n - t_{\ell_0}$ N-isomorph sets converge in an infinite sequence of subdivisions between the cones to $n - t_{\ell_0}$ rays that are vertexed at a point $\bar{x} \in L(\hat{f} - \varepsilon)$. Furthermore, according to Theorem 4.4, by performing subdivisions within the cones the cones shrink to rays. Therefore, the edges of all cones converge to a ray $\bar{x} + \lambda \hat{u}$, $\lambda \geqslant 0$. Since none of the cone edges contains a point $x \in P \setminus L(\hat{f})$, the ray $\bar{x} + \lambda \hat{u}$, $\lambda \geqslant 0$ also contains no point in $P \setminus L(\hat{f})$, i.e. $\{\bar{x} + \lambda \hat{u} \mid \lambda \geqslant 0\} \cap P \subset L(\hat{f})$.

Let $\bar{x} + \hat{\lambda} \hat{u}$ be the intersection point of $\bar{x} + \lambda \hat{u}$, $\lambda \geqslant 0$ with $\mathrm{bd}(L(\hat{f} - \varepsilon))$, and let $\widehat{\varrho} := \hat{u}^{\mathsf{T}}(\bar{x} + \hat{\lambda} \hat{u})$. To derive a decomposition cut we determine the barycenter $\bar{x}^{(\ell)}$ of $S_{t_\ell}^{(\ell)}$ (see (10)) and an 'average' direction $\bar{u}^{(\ell)}$ of the edges of the corresponding cones (see (11)). The depth Δ_ℓ of a decomposition cut $d_\ell^{\mathsf{T}} x \geqslant \delta_\ell$ can be measured as the distance from $\bar{x}^{(\ell)}$ to the point where it intersects the ray $\bar{x}^{(\ell)} + \lambda \bar{u}^{(\ell)}$, $\lambda \geqslant 0$. Note that for $\ell \to \infty$ the ray $\bar{x}^{(\ell)} + \lambda \bar{u}^{(\ell)}$, $\lambda \geqslant 0$ converges to the ray $\bar{x} + \lambda \hat{u}$, $\lambda \geqslant 0$. Hence we can interpret $\hat{u}^{\mathsf{T}} x = \widehat{\varrho}$ as a decomposition cut with depth $\widehat{\Delta} := \hat{\lambda} \|\hat{u}\|$, and it holds

$$\lim_{\ell \to \infty} \Delta_\ell = \widehat{\Delta}. \tag{33}$$

It follows from (33) and from the convergence of the cone edges to the ray $\bar{x}+\lambda\hat{u}$, $\lambda \geqslant 0$, that there exists ℓ_1 with $\ell_1 \geqslant \ell_0$ such that for the decomposition cut $d_\ell^{\mathsf{T}} x \geqslant \delta_\ell$ for $\ell \geqslant \ell_1$ the following hold. First, the decomposition cut $d_\ell^{\mathsf{T}} x \geqslant \delta_\ell$ intersects all edges of the cone $C_{s_\ell}^{(\ell)}(\tilde{x}_i)$, $\tilde{x}_i \in S_{t_\ell}^{(\ell)}$, in $L(\hat{f}-\varepsilon)\setminus L(\hat{f})$. Second, the convex hull of these intersection points is contained in $L(\hat{f}-\varepsilon)\setminus L(\hat{f})$. Since otherwise at least one cone edge would contain a point in $L(\hat{f}-\varepsilon)\setminus L(\hat{f})$, there exists ℓ_2 such that $P^{(\ell)} \subset L(\hat{f})$ for $\ell \geqslant \ell_2$. This implies that the decomposition cut $d_{\hat{\ell}}^{\mathsf{T}} x \geqslant \delta_{\hat{\ell}}$ with $\hat{\ell} := \max\{\ell_1, \ell_2\}$ eliminates $P^{(\hat{\ell})}$ and no further subdivision is due, which is a contradiction. Hence, only finite sequences of subpolytopes are derived, which implies the finiteness of PARTI. □

With Theorem 4.3 and the concepts discussed in the proof of Theorem 5.1 we can prove the following theorem.

THEOREM 5.2. *Let there be the same assumptions as in Theorem* 5.1. *For* $\varepsilon = 0$ PARTI *either terminates at an exact global optimum after a finite number of iterations, or else it involves an infinite sequence of subdivisions. The latter case can occur only if the current best solution is actually globally optimal.*

6. Numerical Experiments

In this section we compare the performance of the partition algorithm, PARTI, with the performance of a pure cutting plane algorithm using decomposition cuts, DECO CUT. Additionally we compare the performance of both with the performance of the well-known conical algorithm, CONICAL. The pure cutting plane algorithm was implemented as proposed in Porembski (1999) and for the conical algorithm we implemented the version proposed in Jaumard and Meyer (1998).

We implemented the algorithms using MatLab 6.1 with Optimization Toolbox. There also exists a toolbox that allows one to convert MatLab code into C/C++ code. The advantage of doing this is that the programs run much faster (by a factor of 10 and more) since C/C++ is a compiled programming language whereas MatLab is an interpreted programming language. Unfortunately, this toolbox was not available to us. The computer used was an 500 MHz Pentium III PC with 256 MB RAM. The operating system was Windows 2000 Professional.

We tested the algorithms on several problems. The following objective functions taken from the literature (cf. Konno 1976b; and Locatelli and Thoai, 2000), have been considered.

$$f_{\text{Konno}}(x) = -2\sum_{j=1}^{n} \xi_j^2 + 2\sum_{j=1}^{n-1} \xi_j \xi_{j+1}$$

$$f_{\text{LT1}}(x) = -3\sum_{j=1}^{n} \xi_j^2 + 2\sum_{j=1}^{n-1} \xi_j \xi_{j+1}$$

$$f_{\mathrm{LT2}}(x) = -\Big(\sum_{j=1}^{n} \xi_j^2\Big) \log\Big(1 + \sum_{j=1}^{n} \xi_j^2\Big).$$

As feasible regions we considered Konno's polytope (cf. Konno, 1976b)

$$P_{\mathrm{Konno}} = \{x \in \mathbb{R}^n \mid A_n x \leqslant b_n,\ x \geqslant 0\},$$

where

$$A_n := \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ n & 1 & \cdots & n-2 & n-1 \end{pmatrix} \quad \text{and} \quad b_n := \frac{n(n+1)}{2} e$$

and e is a vector of n ones. Additionally we consider randomly generated polytopes

$$P_{\mathrm{Bret}} = \{x \in \mathbb{R}^n A_{m,n} x \leqslant b_m,\ e^{\mathsf{T}} x \leqslant n+5,\ x \geqslant 0\},$$

Table 1. Performance of CONICAL, DECO CUT and PARTI

Obj.	Pol.	n	m	Iterations			CPU time		
				CONIC.	DECO C.	PARTI	CONIC.	DECO C.	PARTI
f_{Konno}	P_{Konno}	4	8	11	$4^{(3)}$	1	1.61	2.54	0.57
f_{Konno}	P_{Konno}	5	10	14	$3^{(4)}$	1	3.42	2.95	1.07
f_{Konno}	P_{Konno}	6	12	27	$5^{(5)}$	1	6.64	7.05	2.03
f_{Konno}	P_{Konno}	7	14	65	$8^{(4)}$	2	19.82	16.09	4.18
f_{Konno}	P_{Konno}	8	16	138	$24^{(5)}$	2	121.22	105.25	9.50
f_{Konno}	P_{Konno}	9	18	393	$5^{(8)}$	3	172.33	77.23	29.66
f_{Konno}	P_{Konno}	10	20	732	$4^{(9)}$	5	350.74	149.02	67.05
f_{Konno}	P_{Konno}	11	21	2251	$6^{(9)}$	5	988.66	211.81	131.40
f_{LT1}	P_{Konno}	7	14	55	$3^{(4)}$	2	24.67	3.53	1.73
f_{LT1}	P_{Konno}	8	16	134	$16^{(7)}$	5	36.98	25.23	3.53
f_{LT1}	P_{Konno}	9	18	221	$12^{(7)}$	5	151.32	33.51	6.57
f_{LT1}	P_{Konno}	10	20	516	$6^{(9)}$	6	263.77	125.13	35.81
f_{LT1}	P_{Konno}	11	22	—	$16^{(6)}$	8	—	331.52	150.67
f_{LT2}	P_{Konno}	8	16	97	$5^{(6)}$	2	43.18	6.01	2.81
f_{LT2}	P_{Konno}	9	18	291	$3^{(8)}$	3	145.87	56.58	37.19
f_{LT2}	P_{Konno}	10	20	538	$5^{(8)}$	4	267.66	178.56	89.99
f_{LT2}	P_{Konno}	11	22	—	$6^{(8)}$	5	—	235.58	132.71
f_{LT2}	P_{Konno}	12	24	—	$8^{(6)}$	5	—	283.18	131.76
f_{LT1}	P_{Brett}	7	14	25	$2^{(5)}$	1	5.32	6.58	2.11
f_{LT1}	P_{Brett}	8	16	76	$6^{(6)}$	2	21.98	9.01	5.32
f_{LT1}	P_{Brett}	9	18	156	$8^{(6)}$	4	87.82	26.17	11.10
f_{LT1}	P_{Brett}	10	20	332	$9^{(6)}$	5	211.18	33.67	15.66
f_{LT1}	P_{Brett}	11	22	—	$11^{(6)}$	9	—	56.51	45.99
f_{LT2}	P_{Brett}	8	16	55	$3^{(6)}$	2	23.67	8.67	3.22
f_{LT2}	P_{Brett}	9	18	121	$4^{(6)}$	2	43.67	9.01	5.62
f_{LT2}	P_{Brett}	10	20	189	$6^{(8)}$	2	89.43	53.36	32.17
f_{LT2}	P_{Brett}	11	22	1325	$8^{(8)}$	5	1231.64	99.54	43.61
f_{LT2}	P_{Brett}	12	24	—	$10^{(6)}$	4	—	187.63	82.15

where $A_{m,n} \in \mathbb{R}^{m \times n}$ and $b_m \in \mathbb{R}^m$. To construct $A_{m,n}$ and b_m we follow an approach proposed by Bretthauer and Cabot (1994). Each element α_{ij} of $A_{m,n}$ is a pseudo-random number uniformly distributed on the interval $[-1, 1]$. Each element β_i of b_m is generated as $\beta_i = \sum_{j=1}^{n} |a_{ij}| + 2y_i$, where y_i is a pseudo-random number uniformly distributed on the interval $[0, 1]$. The constraint $e^{\mathsf{T}} x \leqslant n+5$ in P_{Bret} was added to ensure that P_{Bret} is bounded. The accuracy was chosen as $\varepsilon = 10^{-1}$.

The results of the experiments can be found in Table 1. CONIC. and DECO C. in the column heads stand for CONICAL and DECO CUT, respectively. 'Obj.' and 'Pol.' refer to the objective function and the polytope used, 'n' is the number of variables, 'm' is the number of restrictions. The maximal decomposition depth allowed in DECO CUT is indicated by a superscript in parentheses in the first DECO C. column. We allowed the algorithms a maximum time of 7,200 seconds (= 2 hours) to converge. Then the algorithm was terminated, which is indicated in the table by a hyphen.

7. Concluding Remarks

In this paper a new successive partition algorithm for concave minimization is proposed. The algorithm is based on cone decomposition and decomposition cuts, concepts that have already been applied in pure cutting plane algorithms for concave minimization. The basic structure of the proposed algorithm resembles that of conical partition algorithms. Therefore, the algorithm can be extended to a branch-and-bound variant or to a two-phase scheme in a way similar to that used for conical algorithms (see, e.g., Horst and Tuy, 1996).

Acknowledgments

The author is grateful to two anonymous referees for their helpful comments, which led to improvements in the paper.

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